PN quads

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1 Introduction

The paper [VPBM01] proposed a simple construction that, when finely evaluated, provides a visual smoothing effect for coarsely triangulated surface models. Such models appear for example in games. Since the main bottleneck of the graphics hardware at th e turn of the century were the transfer of data from the CPU to the GPU and the pixel shading operations, enabling this fine evaluation in hardware provided an essentially no-cost visual improvement for the user. At the time of writing, the tessellation hardware instrumental to the fine evaluation is expected to go mainstream, i.e. accessible in the major APIs.

I originally implemented PN quads in 1999 to complement my implementation of PN triangles. But PN quads were not included in the final paper [VPBM01]; and the construction seemed too straightforward to be communicated on its own. However, it is good to have a reference, and so below, PN triangles and PN quads are juxtaposed.

Another observation made at the time is that the shape of surfaces can be made more rounded, by taking as input P and N not the mesh vertices and normals (possibly obtained by averaging facet normals), but the limit points and normals of of Catmull-Clark subdivision [CC78]. Even better, when triangles and quads are mixed, limit points of polar subdivision should also be used [MKP07].

2 PN-patches

The indexing of the coefficients (control points) of a patch in total degree three-sided Bernstein-Bézier (BB) form and in four-sided, tensor-product BB form

$$\sum_{i=0}^{3}\sum_{j=0}^{3}\mathbf{n}_{ij}B_i(u)B_j(v), \qquad B_k(t) := \binom{3}{k}(1-t)^{3-k}t^k \tag{1}$$

(see e.g. [Far91]) are shown in Figure 1. We will construct a *geometry patch* $\mathbf{b}$ and a separate *normal patch* $\mathbf{n}$ that is related but not identical to the normal field of the geometry patch.

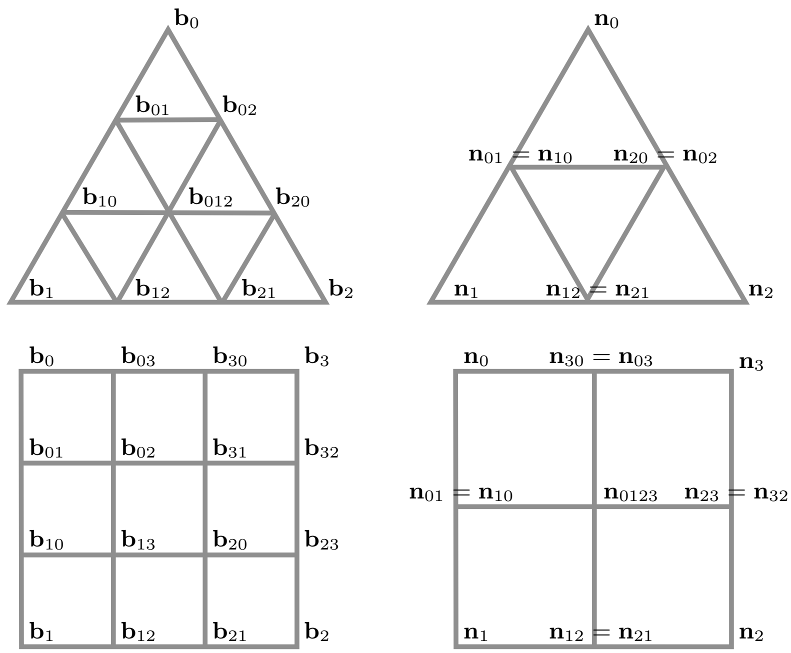

Figure 1: Bézier points of the positional patch (*left*) and the normal patch(*right*). Middle coefficients of the normal patch have two indices for easy enumeration from either corner.

2.1 Boundary Construction.

With the indexing of Figure 1 and denoting the scalar product of two vectors V_1, V_2 by $V_1 \cdot V_2$, we set the coefficients of the boundaries of the patches as

$$\mathbf{b}_i := P_i, \qquad \mathbf{b}_{ij} := \left(2P_i + P_j - \left((P_j - P_i) \cdot N_i\right) N_i\right)/3, \tag{2}$$

$$\mathbf{n}_i := N_i, \qquad \mathbf{n}_{ij} := \frac{\mathbf{h}_{ij}}{\|\mathbf{h}_{ij}\|}, \; \mathbf{h}_{ij} := N_i + N_j - v_{ij}(P_j - P_i), \tag{3}$$

$$v_{ij} := 2\frac{(P_j - P_i) \cdot (N_i + N_i)}{(P_j - P_i) \cdot (P_j - P_i)}.$$

Then the boundary curves of PN-triangle and PN-quad geometry patch $\mathbf{b}$ match the limit points P_i and have tangents orthogonal to the limit normal N_i at the three vertices, $i = 1, 2, 3$. The (bi-)quadratic BB *normal patch* $\mathbf{n}$ matches the N_i at the three vertices and changes according to cubic boundary curve of the geometry patch. Only one of $\mathbf{n}_{ij}$ and $\mathbf{n}_{ji}$ needs to be computed.

Since each boundary depends only on the two endpoints P_i and P_j and their normals N_i and N_j, any two adjacent geometry patches join without gap and also the associated normal patches join continuously. So, unless one examines the silhouettes carefully at the middle of the edge, fine evaluation and rendering using separately the geometry of $\mathbf{b}$ and the normal information from $\mathbf{n}$ gives the appearance of a smooth surface.

2.2 PN-triangles

For each triangle, a three-sided *geometry patch* $\mathbf{b}$ of degree 3 (Figure 1,*top*,*left*) and a normal patch $\mathbf{n}$ of degree 2 (Figure 1,*top*,*right*) are generated. After fixing the boundaries, only the central Bézier coefficient $\mathbf{b}_{012}$ remains to be determined. It is free to choose and is chosen to reproduce quadratics if the input P_i and N_i stem from a quadratic:

$$\mathbf{b}_{012} := (1+\sigma) \sum_{i \neq j, i,j \in \{0,1,2\}} \mathbf{b}_{ij}/6 - \sigma \sum_{i=0}^{2} P_i/3, \qquad \sigma_{\text{default}} := \frac{1}{2}. \tag{4}$$

2.3 PN-quads

For each quad, a four-sided *geometry patch* $\mathbf{b}$ of degree bi-3 (Figure 1,*bottom*,*left*) and a normal patch $\mathbf{n}$ of degree bi-2 (Figure 1,*bottom*,*right*) are generated. We need only set four interior coefficients of $\mathbf{b}$ in a symmetric fashion and one central coefficient of $\mathbf{n}$. Abbreviating $\sum := \sum_{i=1}^{4}$ and interpreting index calculations modulo $n = 4$, we define $\mathbf{q} := \sum(\mathbf{b}_{i,i-1} + \mathbf{b}_{i,i+1})$ as the sum of the eight boundary coefficients that are not corners. Choosing to reproduce bi-quadratics when possible, we obtain

$$\mathbf{b}_{i,i+2} := (1+\sigma)E_i - \sigma V_i, \qquad \sigma_{\text{default}} := \frac{1}{2}, \tag{5}$$

$$18E_i := 2\left(\mathbf{b}_{i,i+1} + \mathbf{b}_{i,i-1} + \mathbf{q}\right) - \left(\mathbf{b}_{i+2,i+1} + \mathbf{b}_{i+2,i-1}\right),$$

$$9V_i := 4P_i + 2(P_{i-1} + P_{i+1}) + P_{i+2},$$

$$\mathbf{n}_{0123} := \left(2\sum \mathbf{n}_{i,i+1} + \sum \mathbf{n}_i\right)/12. \tag{6}$$

The construction can be generalized to multi-sided facets, for example by splitting into triangles and quads.

References

[CC78] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, 10:350–355, 1978.

[Far91] Gerald Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide. Second edition.* Society for Industrial and Applied Mathematics, 1991.

[MKP07] Ashish Myles, Kęstutis Karčiauskas, and Jörg Peters. Extending Catmull-Clark subdivision and PCCM with polar structures. In *PG '07: Proceedings of the 15th Pacific Conference on Computer Graphics and Applications*, pages 313–320, Washington, DC, USA, 2007. IEEE Computer Society.

[VPBM01] Alex Vlachos, Jorg Peters, Chas Boyd, and Jason L. Mitchell. Curved PN triangles. In *2001, Symposium on Interactive 3D Graphics*, Bi-Annual Conference Series, pages 159–166. ACM Press, 2001.