Curve Networks compatible with G^2 surfacing

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Abstract

Prescribing a network of curves to be interpolated by a surface model is a standard approach in geometric design. Where n curves meet, even when they afford a common normal direction, they need to satisfy an algebraic condition, called the vertex enclosure constraint, to allow for an interpolating piecewise polynomial C^1 surface. Here we prove the existence of an additional, more subtle constraint that governs the admissibility of curve networks for G^2 interpolation. Additionally, analogous to the first-order case but using the Monge representation of surfaces, we give a sufficient geometric, G^2 Euler condition on the curve network. When satisfied, this condition guarantees existence of an interpolating surface.

1. Introduction

One much-studied paradigm of geometric design is surface interpolation of a given network of C^2 curve segments (see Figure 1). While many C^2 constructions exist that join n patches (e.g. [Hah89, GH95, Ye97, Rei98, Pra97, YZ04, LS08, KP09]), these constructions *generate* the boundary curves that emanate from the common point, i.e. rely on full control of these curves. In many design scenarios, however, the curves are feature curves. That is, they are *given* and may only be minimally adjusted. It is well-known, that interpolating a network of curves by smooth patches to create a C^1 surface is not always possible when the number of curves is even, since an additional algebraic constraint must hold for the *normal component* of the curve expansion at the common point. This is the *first-order* vertex enclosure constraint [Pet91, DS91, HPS09]. Here we discuss whether curve nets have to meet additional *second-order vertex enclosure constraints* to allow for their G^2 interpolation by smooth surface patches. The two papers on this subject we are aware of are [DS92] which sketches how one might solve the G^2 constraints but does not discuss whether they can be solved and [Pet92] which analyzes the case when curves join with equal angles.

In particular, we want to determine constraints, if any, on n curve segments $\mathbf{y}^j$, $j = 1, \ldots, n$ joining at a vertex so that n sufficiently smooth patches $\mathbf{x}_j$ surrounding the vertex and having $\mathbf{y}^{j-1}$ and $\mathbf{y}^j$ as boundaries can join with C^2 continuity, after reparameterization of the surface patches by some regular smooth maps $\mathbf{\Phi}^j : \mathbb{R}^2 \to \mathbb{R}^2$. The paper analyzes when smooth interpolating surfaces can be constructed. It neither suggests heuristics for the generation of curve networks, nor discusses how to obtain 'fair' surfaces.

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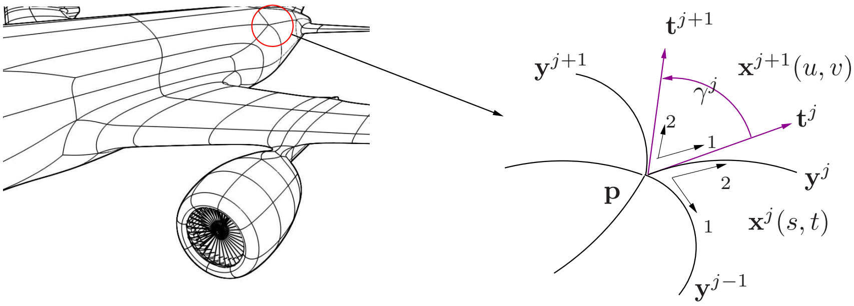

Figure 1: (*left*) Network of curve segments. This paper focuses on (*right*) local network interpolation (see also Definition 1): curves $\mathbf{y}^j$, $j \in \mathbb{Z}_n$, meeting at a point $\mathbf{p}$ are given and pairwise interpolating patches $\mathbf{x}^j$ are sought. The arrow-labels $_1$ and $_2$ indicate the domain parameters associated with the boundary curves of the patches, e.g. $\partial_1 \mathbf{x}^{j+1}(\nu, 0) = \partial_2 \mathbf{x}^j(0, \nu)$.

Overview. Section 2 defines the problem and introduces the notation and the constraints for $\mathtt{k} = 2$ resulting from expanding (7) at $(0, 0)$. Section 3 shows that solvability of the G^2 vertex constraints implies the existence of a solution to the local network interpolation. Section 4 classifies the G^2 constraints at the vertex and analyzes their solvability for a fixed curve network. Theorem 2 establishes the existence of second-order vertex enclosure constraints and therefore of a minimal set of constraints on the curve net. The section ends with a conjecture on the properties of a matrix that holds the key to the complete characterization of second-order vertex enclosure constraints. Section 5 provides a sufficient geometric condition for the existence of a G^2 patch network interpolating a curve network. This is the analogue of the G^1 Euler Condition of [HPS09, HPS10b], but for G^2 networks.

2. Smooth Network Interpolation

As illustrated in Figure 1, we consider n curves $\mathbf{y}^j : \mathbb{R} \to \mathbb{R}^3$ that start at a point $\mathbf{p} \in \mathbb{R}^3$, and we aim at filling-in between the curves using patches $\mathbf{x}^j : \mathbb{R}^2 \to \mathbb{R}^3$, $j \in \mathbb{Z}_n$. In the following we assume that the angle γ^j, between $\mathbf{y}^j$ and $\mathbf{y}^{j+1}$, lies strictly in $(0, \pi)$ [1] We note that the angle γ^j corresponds to patch $\mathbf{x}^{j+1}$ and assume for notational simplicity that the *curves are arclength-parameterized*. In particular, each tangent vector $\mathbf{t}^j := \mathbf{y}_1^j(0)$ is a unit vector. Differential geometry provides us with two fundamental properties that the curve network $\{\mathbf{y}^j\}$ must satisfy to be part of a regular C^2 surface. There must exist a vector $\mathbf{n}$, the normal at $\mathbf{p}$, and $II(\cdot, \cdot)$, the second fundamental form acting on the tangent plane components of its two arguments, such that with the abbreviation $\mathbf{y}_2^j := \partial_2^2 \mathbf{y}^j$,

$$\mathbf{t}^j \cdot \mathbf{n} = 0, \quad \text{and} \quad II(\mathbf{t}^j, \mathbf{t}^j) = \mathbf{y}_2^j \cdot \mathbf{n}, \quad j \in \mathbb{Z}_n. \tag{1}$$

[1] As shown for G^1 continuity [HPS10b], quite different constraints are needed for G^2 interpolation by smooth surface patches when the angle is π or 0.

The existence of a second fundamental form implies the G^1 Euler condition that there exist constants $\kappa_1, \kappa_2 \in \mathbb{R}$ and angles ϕ^j measured from some fixed direction in the tangent plane such that

$$\kappa^j := \mathbf{y}_2^j \cdot \mathbf{n} = \kappa_1 \cos^2 \phi^j + \kappa_2 \sin^2 \phi^j. \tag{2}$$

Just like two linearly independent $\mathbf{t}^j$ define a unique normal $\mathbf{n}$ up to sign, three pair-wise linearly independent $\mathbf{t}^j$ and corresponding normal curvatures can be used to uniquely define a second fundamental form $II(,)$. When the tangents form an X, i.e. when there are just two pair-wise linearly independent directions among the tangents $\mathbf{t}^j$, then there is a one-parameter family of second fundamental forms (cf. (13)) consistent with the curve network.

Definition 1 (Smooth Network Interpolation) *Let*

$$\mathbf{y}^j : \mathbb{R} \to \mathbb{R}^3, \nu \ \mapsto \mathbf{y}^j(\nu), \qquad j \in \mathbb{Z}_n = \{1, \dots, n\} \tag{3}$$

be a sequence of n regular, $C^{2\mathtt{k}}$ continuous curves in $\mathbb{R}^3$ that meet at a common point $\mathbf{p}$ in a plane with oriented normal $\mathbf{n}$ and at angles γ^j less than π (cf. Figure 1):

$$\mathbf{y}^j(0) = \mathbf{p}, \mathbf{t}^j := (\mathbf{y}^j)'(0) \perp \mathbf{n}, \quad 0 < \gamma^j := \angle(\mathbf{t}^j, \mathbf{t}^{j+1}) < \pi. \tag{4}$$

A $G^{\mathtt{k}}$ surface network interpolation *of $\{\mathbf{y}^j\}$ is a sequence of patches*

$$\mathbf{x}^j : \mathbb{R}^2 \to \mathbb{R}^3, (s,t) \mapsto \mathbf{x}^j(s,t), \qquad j \in \mathbb{Z}_n, \tag{5}$$

that are regular and $C^{2\mathtt{k}}$ at $\mathbf{p}$, that interpolate the curve network according to

$$\mathbf{x}^j(\nu, 0) = \mathbf{y}^{j-1}(\nu),\ \mathbf{x}^j(0,\nu) = \mathbf{y}^j(\nu), \tag{6}$$

(with superscript modulo n) and that connect pairwise so that $G^{\mathtt{k}}$ constraints (see e.g. [PBP02] or [Pet02]) hold for $\mathbf{\Phi}^j$

$$\text{at } (u,0) \qquad \partial_1^{k_1}\partial_2^{k_2}\mathbf{x}^{j+1} = \partial_1^{k_1}\partial_2^{k_2}(\mathbf{x}^j \circ \mathbf{\Phi}^j), \quad \text{for } 0 \le k_i \le \mathtt{k}. \tag{7}$$

where $\{\mathbf{\Phi}^j\}_{j\in\mathbb{Z}_n}$ are suitable, say C^{2k} regular maps. Smooth Network Interpolation restricted to the neighborhood of $\mathbf{p}$ is called local network interpolation*.*

Since the reparameterization appears only on one side, the formulation may appear asymmetric; but with $\mathbf{\Phi}^j$ regular, we can invert the relationship – so this formulation is as general and powerful as reparameterizing both $\mathbf{x}^{j+1}$ and $\mathbf{x}^j$. The increased $2\mathtt{k}$th order smoothness at vertices is natural for spline constructions and, intentionally, rules out Gregory's rational constructions [Gre74, MW91, Her96]. Finally, we note that by [HLW99] and regularity, (7) is equivalent to $\partial_2^i\mathbf{x}^{j+1}(u,0) = \partial_2^i(\mathbf{x}^j \circ \mathbf{\Phi}^j)(u,0)$ for $0 \le i \le \mathtt{k}$.

In the following we will focus on smooth network interpolation when $k = 2$. We will assume that the given **curve network is admissible**: that is, every curve of the network is at least C^4 and regular, and the network satisfies (1). We want to characterize when a curvature continuous surface exists that consists of regular C^4 surface patches and interpolates the admissible network.

Notation and constraints. Since our focus is on curvature continuity at $\mathbf{p} = \mathbf{x}^j(0,0)$, we abbreviate the kth derivative of $\mathbf{y}^j$ evaluated at 0 as $\mathbf{y}_k^j$ and write

$$\mathbf{x}_{k_1k_2}^j := (\partial_1^{k_1}\partial_2^{k_2}\mathbf{x}^j)(0,0), \quad \tau_{k_1k_2}^j := (\partial_1^{k_1}\partial_2^{k_2}\tau^j)(0,0), \text{ etc..} \tag{8}$$

We *drop superscripts* whenever the context makes them unambiguous, e.g. we write

$$\mathbf{x}_{k_1k_2} := \mathbf{x}_{k_1k_2}^j,\ \mathbf{x}_{k_1k_2}^- := \mathbf{x}_{k_1k_2}^{j-1}, \ldots, \tag{9}$$

$$\mathbf{y}_k := \mathbf{y}_k^j = \mathbf{x}_{0k},\ \mathbf{y}_k^- := \mathbf{y}_k^{j-1} = \mathbf{x}_{k0},\ \mathbf{y}_k^+ := \mathbf{y}_k^{j+1} = \mathbf{x}_{0k}^+. \tag{10}$$

That is $\mathbf{x}_{k_1k_2}$ is a vector in $\mathbb{R}^3$ and *not* a vector of vectors $[\ldots, \mathbf{x}_{k_1k_2}^j, \ldots]$.

We also *tag the equations arising from* (7) for a specific choice of (k_1, k_2) and j as $(k_1, k_2)^j$. Again, to minimize ink, we leave out the superscript when possible. By (6), $\mathbf{\Phi}^j$ has the expansion

$$\mathbf{\Phi}^j(u,v) := \begin{bmatrix} & (\sigma_{01}^j + \sigma_{11}^j u + \ldots)v & +(\sigma_{02}^j + \sigma_{12}^j u + \ldots)\frac{v^2}{2} + \ldots \\ u+ & (\tau_{01}^j + \tau_{11}^j u + \ldots)v & +(\tau_{02}^j + \tau_{12}^j u + \ldots)\frac{v^2}{2} + \ldots \end{bmatrix}. \tag{11}$$

We call the G^1 and the G^2 constraints labelled $(i,l)^j$ for $i+l \le 4$, i.e. the constraints on the 4-jet of derivatives up to total degree 4 at $\mathbf{p}$, the G^2 *vertex constraints*. Smoothness constraints on the 4-jet suffice to locally characterize the G^2 construction and studying them suffices to determine whether a C^2 surface can be constructed: When satisfied in conjunction with the interpolation constraints, they enable a local network interpolation and this allows for a G^2 surface network interpolation of $\{\mathbf{y}^j\}$ (see Lemma 1). Substituting the curves according to (6), we obtain from (7) at $(0,0)$, via the chain rule, the G^1 constraints

$$\mathbf{y}_1^+ = \mathbf{y}_1^-\sigma_{01} + \mathbf{y}_1\tau_{01} \tag{0,1}$$

$$\mathbf{x}_{11}^+ = \mathbf{y}_1^-\sigma_{11} + \mathbf{x}_{11}\sigma_{01} + \mathbf{y}_2\tau_{01} + \mathbf{y}_1\tau_{11} \tag{1,1}$$

$$\mathbf{x}_{21}^+ = 2\mathbf{x}_{11}\sigma_{11} + \mathbf{y}_1^-\sigma_{21} + \mathbf{x}_{12}\sigma_{01} + \mathbf{y}_3\tau_{01} + 2\mathbf{y}_2\tau_{11} + \mathbf{y}_1\tau_{21} \tag{2,1}$$

$$\begin{aligned}\mathbf{x}_{31}^+ &= 3\mathbf{x}_{12}\sigma_{11} + 3\mathbf{x}_{11}\sigma_{21} + 3\mathbf{y}_1^-\sigma_{31} + \mathbf{x}_{13}\sigma_{01} + \mathbf{y}_4\tau_{01} + 3\mathbf{y}_3\tau_{11} \\ &\quad + 3\mathbf{y}_2\tau_{21} + \mathbf{y}_1\tau_{31}\end{aligned} \tag{3,1}$$

and the G^2 constraints

$$\mathbf{y}_2^+ = \mathbf{y}_2^-\sigma_{01}^2 + 2\sigma_{01}\mathbf{x}_{11}\tau_{01} + \mathbf{y}_1^-\sigma_{02} + \mathbf{y}_2\tau_{01}^2 + \mathbf{y}_1\tau_{02} \tag{0,2}$$

$$\begin{aligned}\mathbf{x}_{12}^+ &= 2\sigma_{11}\mathbf{y}_2^-\sigma_{01} + 2\sigma_{11}\mathbf{x}_{11}\tau_{01} + \mathbf{y}_1^-\sigma_{12} + \mathbf{x}_{21}\sigma_{01}^2 + 2\sigma_{01}\mathbf{x}_{12}\tau_{01} \\ &\quad + \mathbf{x}_{11}\sigma_{02} + \mathbf{y}_3\tau_{01}^2 + \mathbf{y}_2\tau_{02} + 2\tau_{11}\mathbf{x}_{11}\sigma_{01} + 2\tau_{11}\mathbf{y}_2\tau_{01} + \mathbf{y}_1\tau_{12}\end{aligned} \tag{1,2}$$

$$\begin{aligned}\mathbf{x}_{22}^+ &= 2\tau_{21}\mathbf{y}_2\tau_{01} + 4\tau_{11}\mathbf{y}_3\tau_{01} + 4\sigma_{11}\mathbf{x}_{11}\tau_{11} + 2\tau_{21}\mathbf{x}_{11}\sigma_{01} + 4\sigma_{11}\mathbf{x}_{12}\tau_{01} \\ &\quad + 2\sigma_{21}\mathbf{x}_{11}\tau_{01} + 2\sigma_{01}\mathbf{x}_{13}\tau_{01} + 2\sigma_{21}\mathbf{y}_2^-\sigma_{01} + 4\sigma_{11}\mathbf{x}_{21}\sigma_{01} + 2\mathbf{y}_2\tau_{12} \\ &\quad + 4\tau_{11}\mathbf{x}_{12}\sigma_{01} + 2\mathbf{x}_{11}\sigma_{12} + \mathbf{y}_1^-\sigma_{22} + \mathbf{x}_{12}\sigma_{02} + \mathbf{y}_3\tau_{02} + \mathbf{y}_1\tau_{22} \\ &\quad + 2\mathbf{y}_2^-\sigma_{11}^2 + \mathbf{x}_{22}\sigma_{01}^2 + \mathbf{y}_4\tau_{01}^2 + 2\mathbf{y}_2\tau_{11}^2.\end{aligned} \tag{2,2}$$

We start with equations $(i,l)^j$ for $i+l \leq 2$ and check that they are compatible with, and generically implied by (1). Recall that $\mathbf{x}$ is the patch interpolating the curves $\mathbf{y}^-$ and $\mathbf{y}$ with tangents $\mathbf{t}^-$ and $\mathbf{t}$ respectively and $\mathbf{x}^+$ its consecutive patch interpolating $\mathbf{y}$, and $\mathbf{y}^+$ with tangent $\mathbf{t}^+$. First, we derive a second fundamental form II at $\mathbf{p}$ compatible with at least three curves.

Lemma 1 (II derivation) *Equation* (1) *defines* II *unless* $n = 4$ *and the tangents form an X. If the tangents form an X then one additional value* $II(\mathbf{t}^1, \mathbf{t}^2) := w_{11}$ *defines* II.

Proof If the tangents do not form an X then there are three curves with pairwise independent tangents and these define II uniquely (see e.g. [HPS10b, Lemma 1]). Otherwise II is underconstrained and it suffices to specify $II(\mathbf{t}^1, \mathbf{t}^2) := w_{11}$ since $\mathbf{t}^1$ and $\mathbf{t}^2$ are linearly independent. |||

Conversely, II defines the second-order derivatives so that constraints $(i,l)^j$ for $i + l \leq 2$ can be enforced.

Lemma 2 (II and equations $(i,l)^j$ for $i + l \leq 2$) *Given a normal* $\mathbf{n}$ *and a second fundamental form* II *satisfying* (1), *the constraints* $(i,l)^j$ *for* $i + l \leq 2$ *always have a solution.*

Proof Assumption (1), *left*, and regularity imply that constraints of type $(0,1)^j$ hold for some choice of σ_{01} and τ_{01}. In particular $\mathbf{t}^+ = \sigma_{01}\mathbf{t}^- + \tau_{01}\mathbf{t}$. For the remaining equations, we first focus on the normal coordinate. We can enforce the normal component of $(1,1)^j$,

$$\mathbf{n} \cdot \mathbf{x}_{11}^+ = \mathbf{n} \cdot \mathbf{x}_{11}\sigma_{01} + \mathbf{n} \cdot \mathbf{y}_2\tau_{01}, \tag{1,1$_\mathbf{n}$}$$

by setting $\mathbf{n} \cdot \mathbf{x}_{11} := II(\mathbf{t}^-, \mathbf{t})$ and substituting $\mathbf{t}^+ = \sigma_{01}\mathbf{t}^- + \tau_{01}\mathbf{t}$:

$$II(\mathbf{t}, \mathbf{t}^+) = II(\mathbf{t}^-, \mathbf{t})\sigma_{01} + II(\mathbf{t}, \mathbf{t})\tau_{01}. \tag{12}$$

For an X configuration, we see that

$$w_{11} = \mathbf{x}_{11}^0 \cdot \mathbf{n} = \mathbf{x}_{11}^2 \cdot \mathbf{n} = -\mathbf{x}_{11}^1 \cdot \mathbf{n} = -\mathbf{x}_{11}^3 \cdot \mathbf{n}. \tag{13}$$

To verify that the normal components of equations $(0,2)^j$ hold, we take the dot product with the normal $\mathbf{n}$ and apply (1), *right*,

$$\begin{aligned} \mathbf{y}_2^+ \cdot \mathbf{n} &=^{(1)} II(\mathbf{t}^+, \mathbf{t}^+) =^{(0,1)} II(\sigma_{01}\mathbf{t}^- + \tau_{01}\mathbf{t}, \sigma_{01}\mathbf{t}^- + \tau_{01}\mathbf{t}) \\ &= \sigma_{01}^2 II(\mathbf{t}^-, \mathbf{t}^-) + 2\tau_{01}\sigma_{01} II(\mathbf{t}^-, \mathbf{t}) + \tau_{01}^2 II(\mathbf{t}, \mathbf{t}) \\ &= \sigma_{01}^2 \mathbf{y}_2^- \cdot \mathbf{n} + 2\tau_{01}\sigma_{01}\mathbf{x}_{11} \cdot \mathbf{n} + \tau_{01}^2 \mathbf{y}_2 \cdot \mathbf{n}. \end{aligned} \tag{14}$$

The constants τ_{02}^j and σ_{02}^j can be chosen to enforce the tangential coordinates. Similarly, the tangential coordinates of equations $(1,1)^j$ can be enforced by choosing τ_{11}^j and σ_{11}^j. |||

We assume in the following that τ_{ik}^j and σ_{ik}^j for $0 \leq i, k \leq 1$ have been fixed.

3. Constraints on boundary curves arising from G^2 continuity and their sufficiency

First we show that, if we can find a solution satisfying the G^2 vertex constraints then there exists a solution to the local network interpolation. Later, we analyze under what conditions the G^2 vertex constraints are solvable.

Given a network of curves and a solution to the G^2 vertex constraints, we construct a local network interpolation as follows.

Theorem 1 (Sufficiency of G^2 vertex constraints) *If, for some choice of 4-jet of* $\mathbf{x}^j$ *and* $\mathbf{\Phi}^j$*, the* G^2 *vertex constraints hold then there exists a local network interpolation* $\{\mathbf{x}^k\}$.

Proof We drop the superscript k. Assume that the G^2 vertex constraints are satisfied by

$$\mathbf{x}_{ij}, ij \in I, \quad \sigma(u,v) := \sum_{ij \in J} \sigma_{ij} \frac{u^i}{i!} \frac{v^j}{j!}, \quad \tau(u,v) := u + \sum_{ij \in J} \tau_{ij} \frac{u^i}{i!} \frac{v^j}{j!} \tag{15}$$

where $I := \{(1,1),(2,1),(3,1),(1,2),(2,2)\}$, $J := I \cup \{(0,1),(0,2)\}$.

We define a base surface

$$\bar{\mathbf{x}}(s,t) := \mathbf{y}^-(s) + \mathbf{y}(t) - \mathbf{y}_0 + \sum_{ij \in I} \mathbf{x}_{ij} \frac{s^i}{i!} \frac{t^j}{j!} \tag{16}$$

and functions

$$\mathbf{f}_1^+(t) := \frac{\partial}{\partial v}(\bar{\mathbf{x}}(\sigma(u,v),\tau(u,v)) - \bar{\mathbf{x}}^+(u,v))(t,0),$$
$$\mathbf{f}_2^+(t) := \frac{\partial^2}{\partial v^2}(\bar{\mathbf{x}}(\sigma(u,v),\tau(u,v)) - \bar{\mathbf{x}}^+(u,v))(t,0).$$

By definition of $\bar{\mathbf{x}}$ and local network interpolation it can be shown that

$$\frac{d^k \mathbf{f}_1}{dt^k}(t) = \mathbf{0}, 0 \le k \le 3, \qquad \frac{d^k \mathbf{f}_2}{dt^k}(t) = \mathbf{0}, 0 \le k \le 2. \tag{17}$$

We now define the surface patches of the G^2 vertex constraint network as

$$\mathbf{x}(s,t) := \bar{\mathbf{x}}(s,t) + \mathbf{f}_1(s)t + \mathbf{f}_2(s)\frac{t^2}{2}. \tag{18}$$

By construction $\mathbf{x}$ interpolates both $\mathbf{y}^-$ and $\mathbf{y}$ and by definition of $\{\mathbf{f}_1^+, \mathbf{f}_2^+\}$, the conditions for G^2 continuity along the common boundary $\mathbf{y}(t)$ hold,

$$\mathbf{x}_v^+(u,0) = \frac{\partial}{\partial v}(\mathbf{x}(\sigma(u,v),\tau(u,v))(u,0),$$
$$\mathbf{x}_{vv}^+(u,0) = \frac{\partial^2}{\partial v^2}(\mathbf{x}(\sigma(u,v),\tau(u,v))(u,0).$$

Specifically, $\mathbf{f}_1$ and $\mathbf{f}_2$ do not contribute to the partial derivatives of these equations so that the partials of $\mathbf{x}$ can be replaced with those of $\bar{\mathbf{x}}$. |||

Above, by (18), the last two equations of the proof are equivalent to the defining equations of $\mathbf{f}_1$ and $\mathbf{f}_2$. Also $\mathbf{x}(u, v)$ was chosen to make the proof as simple as possible. In general, $\mathbf{x}$ can be defined as

$$\begin{aligned}\mathbf{x}(s,t) &= \mathbf{y}^-(s) + \mathbf{y}(t) - \mathbf{y}_0 + \sum_{ij \in I} \mathbf{x}_{ij} \frac{s^i}{i!} \frac{t^j}{j!} + \\ &\quad \mathbf{f}_1(s)t + \mathbf{f}_2(s)\frac{t^2}{2} + \mathbf{g}_1(t)s + \mathbf{g}_2(t)\frac{s^2}{2} + \mathbf{m}(s,t)\frac{s^3 t^3}{36}\end{aligned}$$

where $\mathbf{m}$ is unrestricted, $\frac{d^k \mathbf{g}_i}{dt^k}(0) = 0, i = 1, 2, k = 0, 1, 2$, and $\{\mathbf{f}_1^+, \mathbf{f}_2^+, \mathbf{g}_1, \mathbf{g}_2\}$ must satisfy constraints generated from the G^2 edge constraints (7). Finally, the reparameterization, Φ, can have higher-order terms.

Now we focus on solvability of the G^s constraints, $s = 0, 1, 2$. The solvability of $(k_1, k_2)^j$ for $k_1 + k_2 = 2$ follows from Lemma 2. Our main goal is therefore to find the local G^s constraints $(k_1, k_2)^j$ of Section 2 for $3 \le k_1 + k_2 \le 4$ in terms of the higher-order derivatives, $\mathbf{x}_{21}^j$, $\mathbf{x}_{12}^j$, $\mathbf{x}_{31}^j$, $\mathbf{x}_{13}^j$, $\mathbf{x}_{22}^j$ and for the reparameterizations' derivatives $\tau_{k_1 k_2}^j$, $\sigma_{k_1 k_2}^j$ for $i, k > 1$. We first consider the equations $(k_1, k_2)^j$ when $k_1 + k_2 = 4$.

Lemma 3 *The equations* $(k_1, k_2)^j$, *where* $k_1 + k_2 = 4$ *and* $1 \le k_1, k_2$ *can always be solved in terms of* $\mathbf{x}_{31}^j$, $\mathbf{x}_{13}^j$, $\mathbf{x}_{22}^j$.

Proof We have more vector-valued variables, $\mathbf{x}_{31}$, $\mathbf{x}_{13}$, $\mathbf{x}_{22}$, than constraints: (3,1), (2,2). Equation (3,1) expresses $\mathbf{x}_{31}^+$ in terms of $\mathbf{x}_{13}$ so that we can focus on solving (2,2) in terms of $\mathbf{x}_{13}$ and $\mathbf{x}_{22}$. Equation (2,2) can be arranged as

$$\mathbf{x}_{22}^+ = \sigma_{01}^2 \mathbf{x}_{22} + 2\tau_{01}\sigma_{01}\mathbf{x}_{13} + f(\mathbf{y}, \mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{21}), \tag{2,2}$$

where $f(\mathbf{y}, \mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{12})$ is the collection of terms on the boundary or appearing in lower-order equations. Clearly, we can solve $n - 1$ of these equations for $\mathbf{x}_{22}^+$. In general, this is all we can hope for since, for equal angles γ^j, the analysis in [Pet92] shows that the constraint matrix for solving (2,2) in terms of just $\mathbf{x}_{22}$ is rank-deficient by 1.

If the tangents do not form an X configuration, i.e. not all consecutive pairs of angles add to π, then at least one $\tau_{01}^j \ne 0$. Let $\tau_{01}^1 \ne 0$. Then, for any choice of $\mathbf{x}_{22}^1$, we can solve (2,2), for $\mathbf{x}_{22}^2, \ldots \mathbf{x}_{22}^n, \mathbf{x}_{13}^1$.

If the tangents form an X then the valence must be $n = 4$ and $\tau_{01} = 0$ and we solve the tangential component for $\mathbf{x}_{22}^2, \ldots \mathbf{x}_{22}^4, \mathbf{x}_{11}^1, \sigma_{12}$ (we may need σ_{12} to choose $w_{11} \ne 0$ for Lemma 6). |||

Our analysis therefore focusses on the case of $k_1 + k_2 = 3$ derivatives. If the corresponding constraints are solvable then no second-order vertex enclosure constraint exists and a construction is always possible. However, the situation is not that simple as the next lemma shows.

Lemma 4 *The equations $(k_1, k_2)^j$, where $k_1 + k_2 = 3$ and $1 \le k_1, k_2$, can be solved in terms of $\mathbf{x}_{12}^j$ and $\mathbf{x}_{21}^j$ if and only if the following $n \times n$ system of equations has a solution:*

$$\mathbf{M}\mathbf{h} = \mathbf{r}, \qquad \mathbf{M}_{jk} := \begin{cases} \sin\gamma, & k = j-1 \\ 2\sin(\gamma^- + \gamma), & k = j \\ \sin\gamma^-, & k = j+1 \\ 0, & else, \end{cases} \tag{19}$$

$$\begin{aligned} \mathbf{r}^j :=& \frac{1}{\sigma_{01}}(2\tau_{01}\sigma_{11} + 2\tau_{11}\sigma_{01} + \sigma_{02})\mathbf{x}_{11} \\ &+ \frac{1}{\sigma_{01}}((\tau_{01})^2\mathbf{y}_3 + 2\tau_{01}\tau_{11}\mathbf{y}_2 + 2\sigma_{01}\sigma_{11}\mathbf{y}_2^- + \tau_{02}\mathbf{y}_2 + \tau_{12}\mathbf{y}_1 + \sigma_{12}\mathbf{y}_1^-) \\ &+ \sigma_{01}(2\sigma_{11}^-\mathbf{x}_{11}^- + \tau_{01}^-\mathbf{y}_3^- + 2\tau_{11}^-\mathbf{y}_2^- + \tau_{21}^-\mathbf{y}_1^- + \sigma_{21}^-\mathbf{y}_1^{j-2}). \end{aligned} \tag{20}$$

In (20), we used the default notation on the right hand side that suppresses the superscript j. We cannot omit the superscript on $\mathbf{r}^j$.

Proof We eliminate $\mathbf{x}_{21}$ by substituting (2,1)$^{j-1}$ into (1,2)^j to obtain

$$\mathbf{x}_{12}^+ = \mathbf{x}_{12}^-\sigma_{01}^-\sigma_{01}^2 + 2\sigma_{01}\mathbf{x}_{12}\tau_{01} + g(\mathbf{y}, \mathbf{x}_{11}), \tag{21}$$

where $g(\mathbf{y}, \mathbf{x}_{11})$ collects the terms depending on $\mathbf{y}$ and $\mathbf{x}_{11}$. We divide both sides by $-\sigma_{01} := \frac{\sin\gamma}{\sin\gamma^-}$ (allowable since by assumption on the opening angles $\sin\gamma \neq 0$) to obtain for $\mathbf{h}^j := -\dfrac{\mathbf{x}_{12}^j}{\sin\gamma^{j-1}}$

$$\sin\gamma^j\mathbf{h}^{j-1} + 2\sin(\gamma^{j-1} + \gamma^j)\mathbf{h}^j + \sin\gamma^{j-1}\mathbf{h}^{j+1} = \frac{1}{\sigma_{01}}g(\mathbf{y}, \mathbf{x}_{11}) =: \mathbf{r}^j. \tag{22}$$

This is Equation (19). |||

Although, generically, we can freely choose all $\tau_{k_1k_2}^j$ and $\sigma_{k_1k_2}^j$ for $k_1 + k_2 > 1$, rank deficiency of the matrix $\mathbf{M}$ could lead to an additional constraint on the boundary curves when we consider a *higher-order saddle point*. For a higher-order saddle point, $\mathbf{n} \cdot \mathbf{y}_k^j = 0$ for $k = 1, 2$ and this can force $\mathbf{n} \cdot \mathbf{x}_{11}^j = 0$ so that

$$\mathbf{n} \cdot \mathbf{r}^j = \frac{(\tau_{01})^2}{\sigma_{01}}\mathbf{n} \cdot \mathbf{y}_3 + \sigma_{01}\tau_{01}^-\mathbf{n} \cdot \mathbf{y}_3^-.$$

If $\ell \in \mathbb{R}^n$ is a left null-vector of $\mathbf{M}$, i.e. $\ell\mathbf{M} = \mathbf{0}$, then we obtain the *second-order vertex enclosure constraint*

$$0 = \ell(\mathbf{r}\mathbf{n}) = \sum_j \tau_{01}^j(\frac{\tau_{01}^j\ell^j}{\sigma_{01}^j} + \sigma_{01}^{j+1}\ell^{j+1})\mathbf{n} \cdot \mathbf{y}_3^j. \tag{23}$$

We therefore focus on the existence of left null-vectors and hence the rank of $\mathbf{M}$. The next lemma partly characterizes $\mathtt{rank}(\mathbf{M})$ and thereby shows in what cases a second-order vertex enclosure constraint *can* exist and where no second-order vertex enclosure constraint exists because $\mathbf{M}$ is of full rank.

Lemma 5 (rank of $\mathbf{M}$) *The rank of $\mathbf{M}$ is at least $n-2$. The matrix $\mathbf{M}$ is of full rank, i.e. $\mathtt{rank}(\mathbf{M}) = n$, if either all angles are equal and $n \notin \{3,4,6\}$; or if all angles are less than $\pi/3$.*

Proof Since all $\sin\gamma^j > 0$, we can solve (22) for $j = 1,\dots,n-2$. Therefore the rank-deficiency is at most 2. Discrete Fourier analysis in [Pet92] shows $\mathbf{M}$ to be of full rank if all angles are equal and $n \notin \{3,4,6\}$. If all angles are less than $\pi/3$ then the matrix is strictly diagonally dominant and therefore invertible. |||

We will see below that, for $n = 4$ and equal angles, $\mathtt{rank}(\mathbf{M}) = 2$; and for $n = 5$, when three angles are $\pi/2$, then $\mathtt{rank}(\mathbf{M}) = 3$. Discrete Fourier analysis in [Pet92] showed for all angles equal that $\mathtt{rank}(\mathbf{M}) = n-1$ if $n \in \{3,6\}$ and $\mathtt{rank}(\mathbf{M}) = n-2$ when $n = 4$. For unequal angles, the analysis is more complex.

As for the first-order vertex enclosure constraint, we can focus exclusively on the normal component of the constraints since, in the tangent plane, we can always choose τ_{11} and σ_{11} to solve (1,1) for an arbitrary choice of $\mathbf{x}_{11}^j$. Then we can use the tangent component of $\mathbf{x}_{11}$ and the free choice of σ_{02} to solve the tangent component of $(22)^1$ and $(22)^n$ (while $(22)^j$ is solved in terms of $\mathbf{x}_{12}^j$, $j = 2,\dots,n-1$). Focussing on the normal component, we note that the right hand side simplifies to

$$\begin{aligned}\mathbf{n}\cdot\mathbf{r}^j =& \frac{1}{\sigma_{01}}(2\tau_{01}\sigma_{11} + 2\tau_{11}\sigma_{01} + \sigma_{02})\mathbf{n}\cdot\mathbf{x}_{11} + 2\sigma_{01}\sigma_{11}^-\mathbf{n}\cdot\mathbf{x}_{11}^- \\ &+\frac{1}{\sigma_{01}}((\tau_{01})^2\mathbf{n}\cdot\mathbf{y}_3 + (2\tau_{01}\tau_{11} + \tau_{02})\mathbf{n}\cdot\mathbf{y}_2) \\ &+\sigma_{01}\tau_{01}^-\mathbf{n}\cdot\mathbf{y}_3^- + 2(\sigma_{11} + \sigma_{01}\tau_{11}^-)\mathbf{n}\cdot\mathbf{y}_2^- .\end{aligned} \tag{24}$$

Lemma 6 *If $n = 3$ or $n = 4$, no second-order vertex enclosure constraint exists for any choice of γ^j.*

Proof For $n = 3$, $\sin(\gamma^- + \gamma) = -\sin\gamma^+$ and $\mathbf{M}$ simplifies to

$$\mathbf{M} = \begin{bmatrix} -2\sin\gamma^2 & \sin\gamma^3 & \sin\gamma^1 \\ \sin\gamma^2 & -2\sin\gamma^3 & \sin\gamma^1 \\ \sin\gamma^2 & \sin\gamma^3 & -2\sin\gamma^1 \end{bmatrix}. \tag{25}$$

Since $0 < \sin\gamma^j \le 1$, multiples of $\ell := [1,1,1]$ are the only null-vectors of $\mathbf{M}$; that is, $\mathtt{rank}(\mathbf{M}) = 2$. We have a solution iff

$$\mathbf{r}^1 + \mathbf{r}^2 + \mathbf{r}^3 = \mathbf{0}. \tag{26}$$

If we choose $\tau_{kl}^j = \sigma_{kl}^j = 0$ for $k + l > 1$ then we have a solution since

$$\begin{aligned}\mathbf{r}^1 + \mathbf{r}^2 + \mathbf{r}^3 &= \sum_{j=1}^{3}\tau_{01}^j\Big(\frac{\tau_{01}^j}{\sigma_{01}^j} + \sigma_{01}^{j+1}\Big)\mathbf{y}_3^j \\ &= \sum_{j=1}^{3}\tau_{01}^j\frac{\sin(\gamma^- + \gamma) + \sin\gamma^+}{-\sin\gamma}\mathbf{y}_3^j = 0.\end{aligned}$$

That is, we can choose $\mathbf{x}_{12}^1$ freely and enforce all $(1,2)^j$ by choice of $\mathbf{x}_{12}^2$ and $\mathbf{x}_{12}^3$. Then $\mathbf{x}_{21}^j$ is uniquely determined by $(2,1)^{j-1}$ and all constraints for $k_1+k_2=3$ hold.

If $n=4$, the determinant of $\mathbf{M}$ is

$$D = \begin{vmatrix} 2\sin(\gamma^4+\gamma^1) & \sin\gamma^4 & 0 & \sin\gamma^1 \\ \sin\gamma^2 & 2\sin(\gamma^1+\gamma^2) & \sin\gamma^1 & 0 \\ 0 & \sin\gamma^3 & 2\sin(\gamma^2+\gamma^3) & \sin\gamma^2 \\ \sin\gamma^3 & 0 & \sin\gamma^4 & 2\sin(\gamma^3+\gamma^4) \end{vmatrix} \tag{27}$$

$$= \left(4\sin(\gamma^4+\gamma^1)\sin(\gamma^1+\gamma^2)+\sin\gamma^1\sin\gamma^3-\sin\gamma^2\sin\gamma^4\right)^2 \tag{28}$$

$$= \left(3\sin\frac{\gamma^1-\gamma^2-\gamma^3+\gamma^4}{2}\sin\frac{\gamma^1+\gamma^2-\gamma^3-\gamma^4}{2}\right)^2 \tag{29}$$

$$= 9\sin^2(\gamma^2+\gamma^3)\sin^2(\gamma^1+\gamma^2). \tag{30}$$

The last equation holds because $\sum\gamma^j=2\pi$. That is $D=0$ if and only if $\gamma^1+\gamma^2=\pi$ and therefore $\gamma^3+\gamma^4=\pi$; or $\gamma^2+\gamma^3=\pi$ and therefore $\gamma^4+\gamma^1=\pi$. That is $D=0$ if and only if at least one pair of tangents, $\mathbf{t}^1$,$\mathbf{t}^3$ or $\mathbf{t}^2$,$\mathbf{t}^4$, is parallel.

If $\gamma^1+\gamma^2=\pi$ and $\gamma^2+\gamma^3=\pi$, i.e. the tangents form an X then $\sin\gamma^j=s$, $j=1,2,3,4$, for some scalar $0<s\le 1$. The matrix

$$\mathbf{M} = \begin{bmatrix} 0 & s & 0 & s \\ s & 0 & s & 0 \\ 0 & s & 0 & s \\ s & 0 & s & 0 \end{bmatrix} \tag{31}$$

is of rank 2 and has left null-vectors $[1,\ -c,\ -1,\ c]$ and $[-c,\ -1,\ c,\ 1]$ for any c, for example $c:=2\cos\gamma^4$. Without loss of generality, we choose $\ell_1:=[1,\ 0,\ -1,\ 0]$ and $\ell_2:=[0,\ -1,\ 0,\ 1]$. Since $\sigma_{01}=1$ and $\tau_{01}=0$ and, by (1,1)$^{j-1}$, $\mathbf{n}\cdot\mathbf{x}_{11}^-=-\mathbf{n}\cdot\mathbf{x}_{11}$

$$\mathbf{n}\cdot\mathbf{r}^j := (2\tau_{11}+\sigma_{02}-2\sigma_{11}^-)\mathbf{n}\cdot\mathbf{x}_{11}+\tau_{02}\mathbf{n}\cdot\mathbf{y}_2+2(\sigma_{11}+\tau_{11}^-)\mathbf{n}\cdot\mathbf{y}_2^-. \tag{32}$$

Choosing, for example, $w_{11}\neq 0$ in (13), we can enforce $\mathbf{n}\cdot\mathbf{r}^j=0$ by choice of σ_{02} and the constraints can be satisfied.

If $\gamma^1+\gamma^2=\pi$ but $\gamma^2+\gamma^3\neq\pi$ then $s_1:=\sin\gamma^2=\sin\gamma^1$ and $s_4:=\sin\gamma^3=\sin\gamma^4$ and hence

$$\mathbf{M} = \begin{bmatrix} 2\sin(\gamma^4+\gamma^1) & \sin\gamma^4 & 0 & \sin\gamma^1 \\ \sin\gamma^1 & 0 & \sin\gamma^1 & 0 \\ 0 & \sin\gamma^4 & -2\sin(\gamma^1+\gamma^4) & \sin\gamma^1 \\ \sin\gamma^4 & 0 & \sin\gamma^4 & 0 \end{bmatrix}. \tag{33}$$

For this $\mathbf{M}$, $\texttt{rank}(\mathbf{M})=3$. Since $\sin(\gamma^4+\gamma^1)=\cos\gamma^4\sin\gamma^1+\cos\gamma^1\sin\gamma^4$,

$$\ell\mathbf{M}=0 \quad \text{for } \ell:=[1,\ -2\cos\gamma^4,\ -1,\ -2\cos\gamma^1]. \tag{34}$$

By (13) one $\mathbf{n}\cdot\mathbf{x}_{11}^j$ can be chosen freely and we can set

$$\ell\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r}^1 - 2\cos\gamma^4\mathbf{n}\cdot\mathbf{r}^2 - \mathbf{n}\cdot\mathbf{r}^3 - 2\cos\gamma^1\mathbf{n}\cdot\mathbf{r}^4 = 0 \tag{35}$$

by judicious choice of σ_{02}^j. |||

Our main result, however, proves that a second-order vertex enclosure constraint exists for a higher valence and some choice of γ. For $n = 5$, we compute

$$\det \mathbf{M} = 18 \prod \sin(\gamma^j + \gamma^{j+1})$$

suggesting an Ansatz with angles $\gamma^1 + \gamma^2 = \pi$.

Theorem 2 (second-order vertex enclosure constraint) *For $n = 5$ a second-order vertex enclosure constraint exists. That is, there exist angles γ^j so that $\mathbf{M}$ has a left null vector ℓ, $\ell\mathbf{M} = 0$; and $\ell\mathbf{r} = 0$ constrains the curves $\mathbf{y}^j$.*

Proof We choose γ^j so that $\gamma^1 + \gamma^2 = \pi$ and all $\mathbf{y}_2^j$ so that $\mathbf{n} \cdot \mathbf{y}_2^j = 0$. A left null vector of $\mathbf{M}$ is

$$\ell := [s_3, 2s_5(c_2 s_3/s_2 + c_3) - s_4, -s_5, s_2, -s_2], \; s_j := \sin\gamma^j, c_j := \cos\gamma^j. \quad (36)$$

Since $\mathbf{n} \cdot \mathbf{y}_2^j = 0$, due to the uniqueness of the solution to the G^1 system, (1) and equations $(1,1)^j$ imply $\mathbf{n} \cdot \mathbf{x}_{11}^j = 0$. Then

$$\begin{aligned} 0 = \ell\mathbf{r} &= \sum_j \tau_{01}^j (\frac{\tau_{01}^j \ell^j}{\sigma_{01}^j} + \sigma_{01}^{j+1}\ell^{j+1})\mathbf{n} \cdot \mathbf{y}_3^j, \qquad s_{j-1,j} := \sin(\gamma^{j-1} + \gamma^j) \\ &= \sum_j \frac{s_{j-1,j}}{-s_j s_{j-1}} (s_{j-1,j}\ell^j + s_{j+1}\ell^{j+1})\mathbf{n} \cdot \mathbf{y}_3^j, \qquad ((23)_5) \end{aligned}$$

has to hold (by assumption on the angles, $s_j > 0$ for all j). Specifically, let us assume that $\mathbf{p}$ is a second-order saddle point, so that $\mathbf{n} \cdot \mathbf{y}_2^j = 0$, and choose $\gamma := \frac{\pi}{6}[3, 3, 2, 2, 2]$. Then

$$\begin{aligned} [\ldots, s_j, \ldots] &= [\,1\;1\;\tfrac{\sqrt{3}}{2}\;\tfrac{\sqrt{3}}{2}\;\tfrac{\sqrt{3}}{2}\,], \; [\ldots, c_j, \ldots] = [\,0\;0\;\tfrac{1}{2}\;\tfrac{1}{2}\;\tfrac{1}{2}\,], \qquad (37) \\ [\ldots, s_{j-1,j}, \ldots] &= [\,\tfrac{1}{2}\;0\;\tfrac{1}{2}\;\tfrac{\sqrt{3}}{2}\;\tfrac{\sqrt{3}}{2}\,], \;\text{ and } \ell = [\,\tfrac{\sqrt{3}}{2}\;0\;\tfrac{\sqrt{3}}{2}\;1\;-1\,]. \end{aligned}$$

The second-order vertex enclosure constraint simplifies to

$$0 = [1, 0, 1, 0, 0][\ldots, \mathbf{n} \cdot \mathbf{y}_3^j, \ldots]^{\mathsf{t}} = \mathbf{n} \cdot \mathbf{y}_3^1 + \mathbf{n} \cdot \mathbf{y}_3^3. \qquad (38)$$

That is, for the two terms corresponding to the curves with opposing tangents, $\mathbf{n} \cdot \mathbf{y}_3^1 = -\mathbf{n} \cdot \mathbf{y}_3^3$ has to hold. |||

The assumption that $\mathbf{x}$ is smooth up to third order at $\mathbf{p}$ is important in the proof. For example, the C^2 surface

$$\begin{cases} (x, y, 0) & x \geq 0, \\ (x, y, x^3) & x < 0, \end{cases}$$

can be partitioned into a G^2 patch network that interpolates the five curves

$$\mathbf{y}^1(t) := (t, 0, 0),\ \mathbf{y}^2(t) := (0, t, 0),\ \mathbf{y}^3(t) := (-t, 0, t^3),$$
$$\mathbf{y}^4(t) := (-t, -t, t^3),\ \mathbf{y}^5(t) := (t, -t, 0).$$

Here $\mathbf{n} \cdot \mathbf{x}_{11} = 0$, $\mathbf{n} \cdot \mathbf{y}_2^1 = 0$ and $\mathbf{n} \cdot (\mathbf{y}_3^1 + \mathbf{y}_3^3) = 6 \neq 0$ seemingly contradicting Theorem 2. However $\mathbf{x}^5$ is not C^3 across the y-axis.

We note that the case $n = 5$ yields a doubly rank-deficient matrix $\mathbf{M}$ when $\gamma = \frac{\pi}{4}[2, 2, 2, 1, 1]$.

4. Higher valences

Theorem 2 established the existence of a second-order vertex enclosure constraint. An explicit proof for valences $n \geq 6$ requires exhibiting the null-vector ℓ and hence a full understanding of the rank of $\mathbf{M}$ in its general form. We have not been able to establish the rank in generality. But we hazard a conjecture.

Conjecture 1 *If, for some j both*

$$|2\sin(\gamma^j + \gamma^{j-1})| < \sin\gamma^j + \sin\gamma^{j-1} \textit{ and} \qquad (39)$$
$$|2\sin(\gamma^j + \gamma^{j-1})| < \sin\gamma^{j+1} + \sin\gamma^{j-2}$$

then there exists a choice of the remaining angles for which $\mathbf{M}$ is rank-deficient.

The conjecture draws on Lemma 5 which proves full rank when $\mathbf{M}$ is diagonally dominant. Equation (39) rules out dominance. The conjecture states that when neither the row nor the column of an index is diagonally dominant then additional angles can be found so that the determinant of $\mathbf{M}$ is zero.

We conclude with a few examples supporting the conjecture. The following choices of n angles γ^j, yield a matrix $\mathbf{M}$ with zero determinant:

Examples supporting Conjecture 1.

n	$[\dots, \gamma^j, \dots] =$	
6	$\frac{\pi}{6}[2, 3, 1, 3, h, 3-h]$,	$h := \frac{6}{\pi}\text{atan}\frac{2\sqrt{3}}{3} \approx 1.636886845$
7	$\frac{\pi}{6}[2, 2, 2, 1, 1, 1, 3]$,	
7	$\frac{\pi}{6}[3, 2, 1, 2, 2, h, 2-h]$,	$h := \frac{6}{\pi}\text{atan}\frac{\sqrt{3}}{29} \approx 0.1139327031$
8	$\frac{\pi}{6}[2, 2, 1, 1, 1, 1, h, 4-h]$,	$h := -\frac{6}{\pi}\text{atan}\frac{483\sqrt{3}}{-147-672\sqrt{6}} \approx 0.8337394914$
12	$\frac{\pi}{12}[4, 1, \dots, 1, h, 11-h]$,	$h \approx 2.237657840$

It is well-known that if the number of curves is even then *generically* G^1 local network interpolation has no solution [HPS09, Pet91, DS91]. That is, in the cases where the curve network admits a solution, small perturbations of the curves deny a solution; and if a network denies a solution there is typically no small perturbation to the curves that allows a solution. We conjecture that in the G^2 case, the situation is more in our favor.

Conjecture 2 *If an admissible curve network does not have a G^2 local network interpolation then there exists a small perturbation to a different admissible curve network that has a G^2 local network interpolation.*

5. Euler Conditions implying existence of a G^2 surface

Analogous, to the G^1 case [HPS10b, Theorem 2], we now exhibit G^2 Euler Conditions that, if they hold, guarantee a solution to the local network interpolation. That is we will exhibit a sufficient condition based only on geometric terms, the tangents, second and a third fundamental form. Since we assume a G^1 Euler condition (2) and since we are interested in the expansion of h^i in a small neighborhood at the common vertex $\mathbf{p}$, we use special local coordinates.

Definition 2 (height function, directional derivative) *Let $\mathbf{x}^i$ be a regular patch of a G^1 interpolation of a network of curves $\mathbf{y}^j$. We choose coordinates so that*

$$\mathbf{x}^i(0,0) := \mathbf{p} = \mathbf{0}, \quad z^i := \mathbf{x}^i \cdot \mathbf{n}, \quad (x^i, y^i) := (\mathbf{x}^i \cdot \mathbf{t}^i, \mathbf{x}^i \cdot \mathbf{b}^i),\ \mathbf{b}^i := \mathbf{n} \times \mathbf{t}^i. \tag{40}$$

Then, locally, by regularity and smoothness in a neighborhood of the origin $\mathbf{0}$, there exists a function $h^i : \mathbb{R}^2 \to \mathbb{R}$, called the height function *of the* Monge representation *of $\mathbf{x}^i$, such that*

$$z^i(u,v) = h^i(x^i(u,v), y^i(u,v)). \tag{41}$$

Due to the increased smoothness of the G^2 interpolation patches in Definition 1 at $\mathbf{p}$, we assume h^i is C^4. The derivative of h with respect to a vector field $\mathbf{d} : \mathbb{R}^2 \to \mathbb{R}^2$ is abbreviated as

$$h_{[\mathbf{d}(x,y)]}(x,y) := \lim_{s\to 0} \frac{h((x,y) + s\mathbf{d}(x,y)) - h(x,y)}{s}. \tag{42}$$

Recursively, we define $h_{[\mathbf{d}_1,\mathbf{d}_2]} := (h_{[\mathbf{d}_1]})_{[\mathbf{d}_2]}$.

We recall that, since the $\mathbf{y}^j$ meeting at $\mathbf{p} = \mathbf{y}^j(0)$ are unit-speed curves,

$$\mathbf{y}_1^j = \mathbf{t}^j, \qquad \mathbf{y}_2^j = \kappa_g^j \mathbf{b}^j + \kappa_n^j \mathbf{n}, \tag{43}$$

where κ_g and κ_n are the *geodesic* and *normal* curvatures of the curve. Denoting the projection of any curve $\mathbf{y}(t)$ of the curve network into the tangent plane as $\varkappa(t)$,

$$\varkappa'(0) = \mathbf{t}, \quad \varkappa''(0) = \kappa_g \mathbf{b}, \quad \text{and } h_{[\varkappa_k]}(\mathbf{0}) = 0 \tag{44}$$

since all first order derivatives of the height function vanish by the choice of coordinates.

Interpolation constraints. If h is the local height function of a local network interpolation then additionally for every curve $\mathbf{y}(t)$ of the curve network

$$\mathbf{y}(t) = [\varkappa(t), z(t)] = [\varkappa(t), h(\varkappa(t))]. \tag{45}$$

That is, the restriction of the graph of the height function to $\mathbb{x}(t)$ matches the curve. Equivalently,

$$h(\mathbb{x}(t)) = z(t). \tag{46}$$

Since we are interested in local network interpolation, we focus on the derivatives of total degree at most 2+2, the 4-jet at $(0, 0)$. By differentiating (46) and applying the product and chain rules, we obtain the $4n$ interpolation constraints on h (recall that we drop superscripts indicating the curve):

$$\begin{aligned} h_{[\mathbb{x}'(t)]}(\mathbb{x}(t)) &= z'(t), & (47)\\ h_{[\mathbb{x}'(t),\mathbb{x}'(t)]}(\mathbb{x}(t)) + h_{[\mathbb{x}''(t)]}(\mathbb{x}(t)) &= z''(t), & (48)\\ h_{[\mathbb{x}'(t),\mathbb{x}'(t),\mathbb{x}'(t)]} + 3h_{[\mathbb{x}'(t),\mathbb{x}''(t)]} + h_{[\mathbb{x}^{(3)}(t)]} &= z'''(t), & (49)\\ h_{[\mathbb{x}'(t),\mathbb{x}'(t),\mathbb{x}'(t),\mathbb{x}'(t)]} + 6h_{[\mathbb{x}'(t),\mathbb{x}'(t),\mathbb{x}''(t)]} & \\ +3h_{[\mathbb{x}''(t),\mathbb{x}''(t)]} + 4h_{[\mathbb{x}'(t),\mathbb{x}^{(3)}(t)]} + h_{[\mathbb{x}^{(4)}(t)]} &= z^{(4)}(t). & (50)\end{aligned}$$

If we evaluate these constraints at $t = 0$ and substitute according to (43) and (44), e.g. all single differentiation terms $h_{[\mathbf{v}]}(0) = 0$ vanish, then we obtain

$$\begin{aligned} h_{[\mathbf{t}]}(\mathbf{0}) &= z'(0) & &= 0, & (47_0)\\ h_{[\mathbf{t},\mathbf{t}]}(\mathbf{0}) &= z''(0) & &= \kappa_n, & (48_0)\\ h_{[\mathbf{t},\mathbf{t},\mathbf{t}]}(\mathbf{0}) + 3\kappa_g h_{[\mathbf{t},\mathbf{b}]}(\mathbf{0}) &= z'''(0) & &= \mathbf{y}_3(0)\cdot\mathbf{n}, & (49_0)\\ h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]}(\mathbf{0}) + 6\kappa_g h_{[\mathbf{t},\mathbf{t},\mathbf{b}]}(\mathbf{0}) & & & & \\ +3\kappa_g^2 h_{[\mathbf{b},\mathbf{b}]}(\mathbf{0}) + 4h_{[\mathbf{t},\mathbb{x}_3]}(\mathbf{0}) &= z^{(4)}(0) & &= \mathbf{y}_4(0)\cdot\mathbf{n}. & (50_0)\end{aligned}$$

$\mathbf{G^2}$ constraints. As a special case of [HLW99], two height functions, h and h^+, meet with G^k continuity along the curve $\mathbf{y}(t)$ if and only if there exists a set of vector fields $\{\mathbf{d}_i(t)\}_{i=1}^k$ nowhere parallel to $\mathbb{x}'(t)$ and for $\ell = 0, \ldots, k$,

$$h_{[\mathbf{d}_1(t),\ldots,\mathbf{d}_\ell(t)]}(\mathbb{x}(t)) = h^+_{[\mathbf{d}_1(t),\ldots,\mathbf{d}_\ell(t)]}(\mathbb{x}(t)). \tag{51}$$

For $k = 2$, near $\mathbf{p}$, we may choose $\mathbf{d}_i(t) = \mathbf{b}$. To cover the 4-jet, we consider the derivatives of h, $h_{[\mathbf{b}]}$ and $h_{[\mathbf{b},\mathbf{b}]}$ up to fourth order, total, in the direction of $\mathbf{t}$. That is, at $t = 0$, using (43) and (44), we obtain

$$\begin{aligned} h: \quad & h_{[\mathbf{t},\mathbf{t}]} &= h^+_{[\mathbf{t},\mathbf{t}]} && (52)\\ & h_{[\mathbf{t},\mathbf{t},\mathbf{t}]} + 3\kappa_g h_{[\mathbf{t},\mathbf{b}]} &= h^+_{[\mathbf{t},\mathbf{t},\mathbf{t}]} + 3\kappa_g h^+_{[\mathbf{t},\mathbf{b}]} && (53)\\ & h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + 6\kappa_g h_{[\mathbf{t},\mathbf{t},\mathbf{b}]} + 4h_{[\mathbf{t},\mathbb{x}_3]} &= h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + 6\kappa_g h^+_{[\mathbf{t},\mathbf{t},\mathbf{b}]} + 4h^+_{[\mathbf{t},\mathbb{x}_3]} && (54)\\ h_{[\mathbf{b}]}: \quad & h_{[\mathbf{t},\mathbf{b}]} &= h^+_{[\mathbf{t},\mathbf{b}]} && (55)\\ & h_{[\mathbf{t},\mathbf{t},\mathbf{b}]} + \kappa_g h_{[\mathbf{b},\mathbf{b}]} &= h^+_{[\mathbf{t},\mathbf{t},\mathbf{b}]} + \kappa_g h^+_{[\mathbf{b},\mathbf{b}]} && (56)\\ & h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{b}]} + 3\kappa_g h_{[\mathbf{t},\mathbf{b},\mathbf{b}]} &= h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{b}]} + 3\kappa_g h^+_{[\mathbf{t},\mathbf{b},\mathbf{b}]} && (57)\\ h_{[\mathbf{b},\mathbf{b}]}: \quad & h_{[\mathbf{t},\mathbf{b},\mathbf{b}]} &= h^+_{[\mathbf{t},\mathbf{b},\mathbf{b}]} && (58)\\ & h_{[\mathbf{t},\mathbf{t},\mathbf{b},\mathbf{b}]} + \kappa_g h_{[\mathbf{b},\mathbf{b},\mathbf{b}]} &= h^+_{[\mathbf{t},\mathbf{t},\mathbf{b},\mathbf{b}]} + \kappa_g h^+_{[\mathbf{b},\mathbf{b},\mathbf{b}]} && (59)\end{aligned}$$

First, we derive the constraint central to the G^1 vertex constraint in terms of the height function.

Lemma 7 ([HPS09]) *For height functions h and h^+ of adjacent surfaces of the G^1 patch network,*

$$h_{[\mathbf{t}^-,\mathbf{t}]}\sin\gamma + h^+_{[\mathbf{t},\mathbf{t}^+]}\sin\gamma^+ = \kappa_n \frac{\sin(\gamma+\gamma^+)}{\sin\gamma\sin\gamma^+}. \tag{60}$$

Since $h_{[\mathbf{t}^-,\mathbf{t}]}$ is the normal component of the corner twist vector for the surface patch $\mathbf{x}$, (60) equals [HPS10b, Constraint (17)].

Proof By adjacency, h and h^+ interpolate the curve pairs $(\mathbf{y}^-,\mathbf{y})$ and $(\mathbf{y},\mathbf{y}^+)$ respectively. Since $\mathbf{t}^- = \cos\gamma\mathbf{t} - \sin\gamma\mathbf{b}$, $\mathbf{t}^+ = \cos\gamma^+\mathbf{t} + \sin\gamma^+\mathbf{b}$ and $\sin\gamma \neq 0$ for $\pi > \gamma > 0$,

$$\mathbf{b} = \frac{-\mathbf{t}^- + \cos\gamma\mathbf{t}}{\sin\gamma} = \frac{\mathbf{t}^+ - \cos\gamma^+\mathbf{t}}{\sin\gamma^+}. \tag{61}$$

Then in (55), we can replace differentiation in the direction of $\mathbf{b}$ with differentiation in a linear combination of curve tangents:

$$\frac{\cos\gamma}{\sin\gamma}h_{[\mathbf{t},\mathbf{t}]} - \frac{1}{\sin\gamma}h_{[\mathbf{t},\mathbf{t}^-]} = -\frac{\cos\gamma^+}{\sin\gamma^+}h^+_{[\mathbf{t},\mathbf{t}]} + \frac{1}{\sin\gamma^+}h^+_{[\mathbf{t},\mathbf{t}^+]}. \tag{62}$$

The claim follows since, by (1), $h^+_{[\mathbf{t},\mathbf{t}]} = h_{[\mathbf{t},\mathbf{t}]} = \kappa$. |||

We observe that, by Lemma 1, the entries of second fundamental form of the surface $\mathbf{x}$ at the vertex $\mathbf{p}$ and with principle directions $\mathbf{t}$ and $\mathbf{b}$ are $h_{[\mathbf{t},\mathbf{t}]}$, $h_{[\mathbf{t},\mathbf{b}]}$, $h_{[\mathbf{b},\mathbf{b}]}$.

Next, we postulate that in addition to (1) there is a trilinear form $III(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$ that satisfies (49_0) for all curves $\mathbf{y}^i(t)$. The existence of this form III implies solvability of the equations (k_1,k_2) with $k_1+k_2=3$ that were the focus of attention due to Lemma 4. The next theorem confirms that the existence of such a trilinear form implies the existence of a G^2 surface network.

Theorem 3 ($\mathbf{G^2}$ Euler condition) *Let the G^1 Euler condition hold, i.e. there exists a symmetric bilinear form, II, such that $II(\mathbf{t}^i,\mathbf{t}^i) = \mathbf{y}_2^i \cdot \mathbf{n} = \kappa_n^i$ for all i. If there exists a symmetric trilinear form, III, such that for all i*

$$III(\mathbf{t}^i,\mathbf{t}^i,\mathbf{t}^i) = \mathbf{y}_3^i \cdot \mathbf{n} - 3\kappa_g^i II(\mathbf{t}^i,\mathbf{b}^i) \tag{63}$$

then the curve network $\{\mathbf{y}^i(t)\}$ has a G^2 patch network.

Proof We define the pieces h^i of a height function for each patch $\mathbf{x}^i$ up to third order by

$$h^i(0,0) := 0, \quad h^i_{[\mathbf{v}_1]}(0,0) := 0, \quad h^i_{[\mathbf{v}_1,\mathbf{v}_2]}(0,0) := II(\mathbf{v}_1,\mathbf{v}_2),$$
$$h^i_{[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]}(0,0) := III(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3).$$

By the G^1 Euler condition and (63), the interpolation constraints (48) and (49) hold and the smoothness constraints (52), (53), (55), (56) and (58) hold. Therefore, we need only focus on the derivatives of total degree 4, which, dropping the superscript and evaluating at $(0,0)$ simplify to the $3n$ constraints

$$h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]}, \tag{54_1}$$

$$h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{b}]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{b}]}, \tag{57_1}$$

$$h_{[\mathbf{t},\mathbf{t},\mathbf{b},\mathbf{b}]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{b},\mathbf{b}]}. \tag{59_1}$$

To show that this system always has a solution, we use that by regularity,

$$\mathbf{b} = \alpha\mathbf{t} + \beta\mathbf{t}^+, \qquad \beta \neq 0, \quad (\alpha := \alpha^i, \beta = \beta^i). \tag{64}$$

We can set $h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]}$ and $h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]}$ to enforce the interpolation constraints (50) and (54_1). The remaining $2n$ constraints,

$$\alpha h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + \beta h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]} = \alpha h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + \beta h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]}, \tag{57_2}$$

$$\alpha^2 h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + 2\alpha\beta h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]} + \beta^2 h_{[\mathbf{t},\mathbf{t},\mathbf{t}^+,\mathbf{t}^+]} = \alpha^2 h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + 2\alpha\beta h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]} + \beta^2 h^+_{[\mathbf{t},\mathbf{t},\mathbf{t}^+,\mathbf{t}^+]}, \tag{59_2}$$

simplify due to (54_1) and since $\beta \neq 0$ to

$$h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]}, \tag{57_3}$$

$$h_{[\mathbf{t},\mathbf{t},\mathbf{t}^+,\mathbf{t}^+]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{t}^+,\mathbf{t}^+]}. \tag{59_3}$$

Since also $\mathbf{t}^+ = \tau\mathbf{t} + \sigma\mathbf{t}^-$ and $\sigma \neq 0$ by regularity, we can expand the 4-linear forms to

$$\tau h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + \sigma h_{[\mathbf{t}^-,\mathbf{t},\mathbf{t},\mathbf{t}]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]}, \tag{57_4}$$

$$\tau^2 h_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}]} + 2\tau\sigma h_{[\mathbf{t}^-,\mathbf{t},\mathbf{t},\mathbf{t}]} + \sigma^2 h_{[\mathbf{t}^-,\mathbf{t}^-,\mathbf{t},\mathbf{t}]} = h^+_{[\mathbf{t},\mathbf{t},\mathbf{t}^+,\mathbf{t}^+]}. \tag{59_4}$$

Evidently, the equations of type (57_4) can be enforced by setting $h^+_{[\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}^+]}$ and we can proceed analogous to Lemma 3. If one, say τ_1 is not zero then we can solve all but one of the equations of type (59_4) for $h_{[\mathbf{t}^-,\mathbf{t}^-,\mathbf{t},\mathbf{t}]}$ and the remaining one for $h^1_{[\mathbf{t}^n,\mathbf{t}^1,\mathbf{t}^1,\mathbf{t}^1]}$. If all $\tau = 0$ then we have the X configuration and the valence is 4 and $\sigma = 1$. The system is underconstrained and can be solved by fixing, say $h^1_{[\mathbf{t}^n,\mathbf{t}^n,\mathbf{t}^1,\mathbf{t}^1]}$ and solving for the other three expressions of type $h_{[\mathbf{t}^-,\mathbf{t}^-,\mathbf{t},\mathbf{t}]}$.

This solution to the G^2 constraints on the Monge form provides a solution to the component of the full G^2 vertex constraints in the normal direction at $\mathbf{p}$. By the argument preceding (24), this solution can be completed, in the two tangential directions, to a solution of the full G^2 vertex constraints. Therefore Theorem 1 can be invoked to provide a local construction of the surface. |||

6. Conclusion

We established the existence of a second-order vertex enclosure constraint that governs what admissible curve networks allow for G^2 interpolation by smooth patches. While the first-order vertex enclosure constraint strongly restricts all even-patch configurations, the second-order vertex enclosure constraint is a more subtle constraint. Most input curve meshes that satisfy the first-order vertex enclosure constraint can be expected to also satisfy the second-order vertex enclosure constraint and only a minority will not.

We fully analyzed the practically important cases of valence 3,4 and 5 and characterized the second-order vertex enclosure constraint for valence 5. All other cases still lack a complete characterization of the null-space of $\mathbf{M}$. Lemma 5 establishes bounds on the angle distribution that guarantee that a curve network has a local network interpolation. Conversely, we showed that a solution to the G^2 vertex constraints allows constructing a G^2 local network interpolation.

We used the Monge representation and directional derivatives of the local height function to show that a geometric rather than an algebraic G^2 Euler condition, in terms of second and third order multi-linear forms, suffices to guarantee the existence of a G^2 local network interpolation. This notation allows in principle to investigate higher, $\mathtt{k}$th-order interpolation and smoothness constraints. But, although patterns emerge in constraints analogous to (47_0) through (50_0) and (52) through (59), it is not clear that for $\mathtt{k} > 2$, Euler conditions in terms of the second to the $\mathtt{k}+1$st order multi-linear forms suffice to ensure solvability of constraints involving the $2\mathtt{k}$ jets of the surfaces meeting at the central point.

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