

On the Complexity of Smooth Spline Surfaces from Quad Meshes

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Abstract

This paper derives strong relations that *boundary curves* of a smooth complex of patches have to obey when the patches are computed by local averaging. These relations restrict the choice of reparameterizations for geometric continuity.

In particular, when one *bicubic tensor-product B-spline patch* is associated with each facet of a quadrilateral mesh with n -valent vertices and we do not want segments of the boundary curves forced to be linear, then the relations dictate the *minimal number and multiplicity of knots*: For general data, the tensor-product spline patches must have at least two internal double knots per edge to be able to model a G^1 -connected complex of C^1 splines. This lower bound on the complexity of any construction is proven to be sharp by suitably interpreting an existing surface construction. That is, we have a tight bound on the complexity of smoothing quad meshes with bicubic tensor-product B-spline patches.

1 Introduction

Even though every newly proposed smooth surface construction seeks to be optimal in some aspect, the overall theory of smooth surface constructions offers few sharp lower bounds, i.e. proofs that no polynomial construction of lower degree is possible and that a construction of this least degree exists so that upper bound and lower bound match. One well-appreciated bound is the degree-6 bound for C^2 subdivision surfaces derived by Reif and Prautzsch [Rei96, Pra97] and shown to be sharp, for example by [Rei98, PR99]. Such sharp bounds allow us to understand the fundamental difficulty of the task, and to guide future research by showing where research is futile and what assumptions must be side-stepped to derive substantially new results (see e.g. [MP09]).

We are motivated by a standard task of geometric design: to determine G^1 -connected tensor-product B-spline patches approximating a quadrilateral mesh whose vertices can have any fixed valence. While this challenge can be met by recursive subdivision [CC78], representing the surface with a finite small

number of patches defined by the quad and its neighbors is often preferable, for example to parallelize the construction (see e.g. [LS08, MYP08]). This raises the question: (Q) *what is the simplest structure (in distribution and number of knots) of degree bi-3 spline patches that allow a quad mesh to be converted by localized operations into a smooth surface with one spline patch per quad?* Surprisingly, this basic question has not been settled to date.

To frame the question, Section 2 takes a more general view. We do not constrain the domain to be a collection of quadrilaterals or the functions to be polynomial splines. Also, the relations in Lemmas 1, 2 and 3 do not depend on locality of the construction but apply to any collection of sufficiently smooth patches coming together with a logically symmetric G^1 join: $\partial_2 \mathbf{b}^k(u, 0) + \partial_1 \mathbf{b}^{k-1}(0, u) = \alpha^k(u) \partial_1 \mathbf{b}^k(u, 0)$ (see Definition 1, page 4). Adding locality of operations as a requirement in Section 2.1 then rules out everywhere (piecewise) linear α^k , still in a very general setting.

In Section 3, we specialize the setting to polynomial tensor-product splines of degree bi-3. For these, we obtain a lower bound on the number and multiplicity of knots. We prove that at least two internal double knots are required per edge to admit a local construction. This lower bound is tight, because the recently-published construction for smooth surfaces [FP08] can be re-interpreted as a spline construction with exactly two internal double knots. Together, the lower and upper bound answer the question Q.

1.1 Bi-3 constructions in the literature

Creating C^1 surfaces with a finite number of patches of degree bi-3, i.e. generalizing standard tensor-product B-splines to smooth surfaces from arbitrary manifold quad meshes, is a classic challenge of CAGD (see e.g. [Bez77, vW86, Pet91b]). The assumption that a simple construction with a finite number of patches is not possible motivates Catmull-Clark subdivision (Fig. 1, *left*). PCCM [Pet00] is a finite construction that approximates Catmull-Clark limit surfaces with smoothly connected bi-3 patches. PCCM requires up to two steps of Catmull-Clark subdivision to separate non-4-valent vertices. This proves that a 4×4 arrangement of polynomial patches per quad suffices in principle, corresponding to two double interior knots and one single knot (Fig. 1, *middle*). However, PCCM can have poor shape for certain higher-order saddles (Fig. 5, [URL, Pet01, LS08]). More recently, a number of papers appeared that are predicated on the assumption that a simple construction with a finite number of patches is not possible. Shi et al. [SWWL04, SLW06] propose a subdivision-like refinement approach with bi-3 tensor-product patches to obtain C^0 surfaces where ever more single knots are inserted. They correctly surmise that, in general, no finite C^1 construction with C^2 tensor-product splines of degree bi-3 is possible (see Theorem 1 of our paper). At the other extreme, using a single patch per quad, Loop and Schaefer [LS08] propose a bi-3 C^0 surface construction with separate tangent patches to convey an impression of smoothness as in [VPBM01], while Myles *et al.* [MYP08] perturb a bi-3 base patch near non-4-valent vertices to obtain a C^1 surface of degree bi-5 for CAD applications.

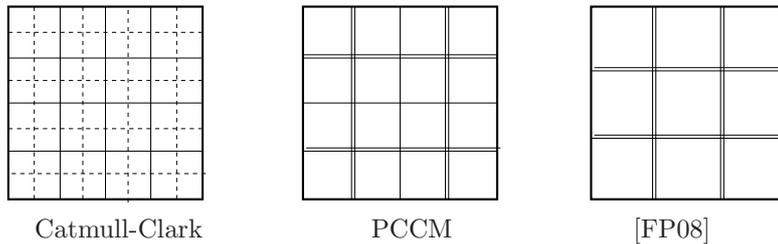


Figure 1: **Knot distribution.** A quadrilateral piece generated by Catmull-Clark subdivision has (infinitely many) single knots, a piece of PCCM requires two double and at least one more single knot, and the construction [FP08] has two double interior knots (which this paper shows to be the minimal number of knots).

Hahmann et al. [HBC08] propose a 2×2 macro-patch per quad; and Fan and Peters [FP08] present an algorithm that constructs smoothly connected Bézier patches of degree bi-3 whose internal transitions allow re-interpretation as one tensor-product spline patch per quad with two internal double knots (Fig. 1, *right*, Corollary 4). We will see that this is indeed the minimal number and multiplicity of knots for the standard Catmull-Clark layout of patches. The structurally different polar layout allows collapsed bi-3 spline patches with single internal knots to complete a C^1 surface [MKP07].

2 Unbiased G^1 constraints

We consider n parameterically C^1 patches

$$\mathbf{b}^k : \square \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad k = 1, \dots, n \quad (1)$$

meeting at a central point $\mathbf{b}^k(0,0) = \mathbf{p}$ such that $\mathbf{b}^k(u,0) = \mathbf{b}^{k-1}(0,u)$ (see Fig. 2). In Section 3, we will assume that \square is the unit square. For now, we only assume that the origin is a corner of the domain \square and that exactly two bounding edges, \mathbf{e}_1 with endpoint $(1,0)$ and \mathbf{e}_2 with endpoint $(0,1)$, start from $(0,0)$. That is, the results of this section also apply, say, to m -sided patches. We also assume that the patches are not singular at the origin in the sense that $\partial_2 \mathbf{b}^k(0,0) \times \partial_1 \mathbf{b}^k(0,0) \neq 0$ where ∂_ℓ denotes differentiation with respect to the ℓ th argument. That is, we do not here consider singular patches such as constructed in [Pet91a, NP94, Rei98].

To make the n patches form a C^1 surface, we want to enforce logically symmetric (unbiased) G^1 constraints. (We will discuss the general case in Section 4.)

Definition 1 (Unbiased G^1 constraints) *With $\alpha^k : \mathbb{R} \rightarrow \mathbb{R}$ a sufficiently smooth, univariate scalar-valued function, the unbiased G^1 constraints between*

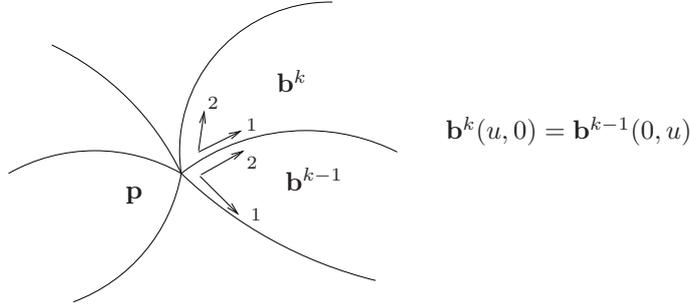


Figure 2: **Indexing and parameterization** of adjacent patches at a vertex of valence n (if $k = 1$ then $\mathbf{b}^{k-1} = \mathbf{b}^n$), illustrating the G^1 constraints (2) .

consecutive patches are

$$\partial_2 \mathbf{b}^k(u, 0) + \partial_1 \mathbf{b}^{k-1}(0, u) = \alpha^k(u) \partial_1 \mathbf{b}^k(u, 0). \quad (2)$$

If $\alpha^k \equiv 0$, the constraints enforce parametric C^1 continuity.

We abbreviate

$$a^{k,\ell} \in \mathbb{R}, \quad \text{the } \ell\text{th derivative of } \alpha^k \text{ evaluated at } 0 \quad (3)$$

and the tangent

$$\mathbf{t}^k := \partial_1 \mathbf{b}^k(0, 0) \in \mathbb{R}^3 \quad (4)$$

so that relation (2) becomes at $(0, 0)$

$$\mathbf{t}^{k+1} + \mathbf{t}^{k-1} = a^{k,0} \mathbf{t}^k. \quad (2)_{u=0}$$

That is, the first superscript counts sectors surrounding $(0, 0)$ modulo n while the second indicates derivatives.

We now mimic the setup for spline patches by assuming higher differentiability in the vicinity of the intersection of the boundary with a family of line segments that partition the domain. Since we do not insist on an orthogonal grid of knot lines, or a rectangular domain, we will call such smooth functions generalized splines.

Definition 2 (Knot lines, edge knots and generalized splines) A C^s generalized spline patch is a map $\mathbf{b}^k : \square \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ that is $s > 0$ times continuously

differentiable. An edge knot is one of a finite number of points $(t_j, 0)$ on \mathbf{e}_1 , respectively $(0, t_j)$ on \mathbf{e}_2 with $0 < t_j < 1$. Exactly one line segment, called knot line, starts at each edge knot into the interior of \square . At every edge knot, on either side of its knot line,

$$\partial_2 \mathbf{b}^k \times \partial_1 \mathbf{b}^k \neq 0, \quad \partial_1^i \partial_2^j \mathbf{b}^k \text{ is well-defined for } i + j \leq s + 1, \quad (5)$$

$$\text{and} \quad \partial_1^i \partial_2^j \mathbf{b}^k = \partial_2^j \partial_1^i \mathbf{b}^k. \quad (6)$$

The generalized spline definition is intentionally broader than its subclass of polynomial tensor-product splines that motivates it. It includes other piecewise constructs with C^s smooth transitions, for example, trigonometric splines, rational multi-sided patches or subdivision constructions. Note that an edge knot and therefore a knot line can have ‘multiplicity’ greater than 1 when viewed in the standard spline setting.

For C^1 generalized splines, we can then differentiate relation (2) along (the respective domain edge of) the common boundary $\mathbf{b}^k(u, 0) = \mathbf{b}^{k-1}(0, u)$:

$$\begin{aligned} & (\partial_1 \partial_2 \mathbf{b}^k)(u, 0) + (\partial_2 \partial_1 \mathbf{b}^{k-1})(0, u) \\ &= \alpha^k(u) \partial_1^2 \mathbf{b}^k(u, 0) + (\alpha^k)'(u) \partial_1 \mathbf{b}^k(u, 0). \end{aligned} \quad (7)$$

When we evaluate at $u = 0$ then

$$\text{at } (0, 0), \quad \partial_1 \partial_2 \mathbf{b}^k + \partial_2 \partial_1 \mathbf{b}^{k-1} = a^{k,0} \partial_1^2 \mathbf{b}^k + a^{k,1} \partial_1 \mathbf{b}^k. \quad (8)$$

If n is *even* then the alternating sum of the left hand sides vanishes

$$\text{at } (0, 0), \quad \sum_{k=1}^n (-1)^k (\partial_1 \partial_2 \mathbf{b}^k + \partial_2 \partial_1 \mathbf{b}^{k-1}) = 0 \quad (9)$$

and therefore so must the right hand side

$$\text{at } (0, 0), \quad 0 = \sum_{k=1}^n (-1)^k a^{k,0} \partial_1^2 \mathbf{b}^k + \sum_{k=1}^n (-1)^k a^{k,1} \partial_1 \mathbf{b}^k. \quad (10)$$

In particular, if the patches join smoothly and therefore have a unique normal $\mathbf{n} \in \mathbb{R}^3$ at \mathbf{p} then, with \cdot denoting the Euclidean inner product,

$$\text{if } n \text{ is even, at } (0,0) \quad 0 = \sum_{k=1}^n (-1)^k a^{k,0} \mathbf{n} \cdot \partial_1^2 \mathbf{b}^k. \quad (11)$$

This is the *vertex-enclosure constraint* (see e.g. [Pet02, p.205],[HPS09]).

We briefly focus on the important generic case where $n = 4$ patches meet.

Definition 3 (tangent X) *If $n = 4$, $\partial_1 \mathbf{b}^1(0, 0) = -\partial_1 \mathbf{b}^3(0, 0)$ and $\partial_1 \mathbf{b}^2(0, 0) = -\partial_1 \mathbf{b}^4(0, 0)$ then the tangents form an X.*

Lemma 1 (X tangent) *If the tangents of four C^1 generalized splines form an X, then $a^{1,1} = a^{3,1}$ and $a^{2,1} = a^{4,1}$.*

Proof If the tangents form an X then $n = 4$ and $a^{k,0} = 0$, $k = 1, 2, 3, 4$ so that (10) simplifies to

$$\text{at } (0,0), \quad 0 = (a^{1,1} - a^{3,1})\partial_1 \mathbf{b}^1 - (a^{2,1} - a^{4,1})\partial_1 \mathbf{b}^2. \quad (12)$$

Since the patches are regular at corners, both summands have to vanish, implying the claim. \parallel

We now consider the unbiased G^1 transition between two C^1 generalized spline patches. We focus on an *edge vertex*, the image of an edge knot, that is not an endpoint of the boundary. Without loss of generality, we can assume that at such an edge vertex four pieces meet such that \mathbf{b}^1 and \mathbf{b}^2 belong to one generalized spline patch, and \mathbf{b}^3 and \mathbf{b}^4 are pieces of the edge-adjacent generalized spline patch (Figure 3). For, if an edge knot does not have a counterpart in the neighboring generalized spline patch, i.e. we have a ‘T-corner’, we can add an edge knot and a corresponding knot line to subdivide the neighboring patch. Since each generalized spline patch is internally parameterically C^1 , by Definition 1

$$\alpha^2 \equiv 0 \equiv \alpha^4. \quad (13)$$

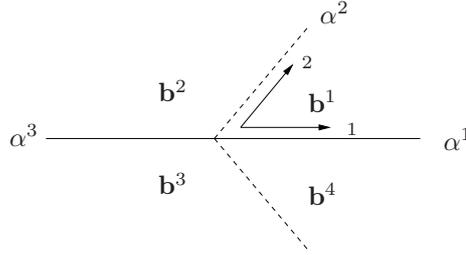


Figure 3: Join across an **edge knot** on the boundary (solid) between two generalized splines. The first generalized spline has pieces \mathbf{b}^1 and \mathbf{b}^2 .

Lemma 2 (C^1 generalized spline, edge vertex) *Let $(0,0)$ be the parameter associated with an edge vertex on the boundary common to two C^1 generalized splines that are joined by unbiased G^1 constraints. Then*

$$a^{1,0} = -a^{3,0}, \quad (14)$$

$$\text{at } (0,0) : 0 = a^{1,0}(\partial_1^2 \mathbf{b}^1 - \partial_1^2 \mathbf{b}^3) + (a^{1,1} - a^{3,1})\mathbf{t}^1. \quad (15)$$

Proof Since $n = 4$, $a^{1,0}\mathbf{t}^1 = \mathbf{t}^2 + \mathbf{t}^4 = a^{3,0}\mathbf{t}^3$ and the parametric C^1 constraints imply $\mathbf{t}^1 = -\mathbf{t}^3$ so that (14) follows. By (13), (10) specializes to

$$\begin{aligned} \text{at } (0,0), \quad 0 &= a^{1,0}\partial_1^2 \mathbf{b}^1 + a^{3,0}\partial_1^2 \mathbf{b}^3 + a^{1,1}\partial_1 \mathbf{b}^1 + a^{3,1}\partial_1 \mathbf{b}^3 \\ &= a^{1,0}(\partial_1^2 \mathbf{b}^1 - \partial_1^2 \mathbf{b}^3) + (a^{1,1} - a^{3,1})\mathbf{t}^1 \end{aligned}$$

as claimed. |||

So, remarkably, when two generalized spline patches meet along a common boundary, unbiased G^1 constraints *across* this boundary imply the constraint (15) exclusively in terms of derivatives *along* the boundary.

Lemma 3 (C^2 generalized spline, edge vertex) *Let $(0, 0)$ be the parameter associated with an edge vertex of the boundary common to two C^2 generalized splines joined by unbiased G^1 constraints. Then, in addition to (14), at $(0, 0)$,*

$$a^{1,1} = a^{3,1}, \quad (16)$$

$$0 = a^{1,0}(\partial_1^3 \mathbf{b}^1 - \partial_1^3 \mathbf{b}^3) + 4a^{1,1} \partial_1^2 \mathbf{b}^1 + (a^{1,2} - a^{3,2}) \mathbf{t}^1. \quad (17)$$

Proof Since the generalized splines are C^2 , $\partial_1^2 \mathbf{b}^1(0, 0) = \partial_1^2 \mathbf{b}^3(0, 0)$. Then (15) is equivalent to (16).

Parametric C^2 continuity across the spline-internal boundaries (see dashed lines in Fig. 3) implies

$$\text{for } k = 2, 4, \text{ at } (0, 0), \quad \partial_2 \partial_1 \partial_2 \mathbf{b}^k + \partial_1 \partial_2 \partial_1 \mathbf{b}^{k-1} = 0. \quad (18)$$

Differentiating (7) once more along the (direction corresponding to the) common boundary of the two generalized splines, we obtain for $k = 1, 3$, at $(0, 0)$,

$$\partial_1 \partial_1 \partial_2 \mathbf{b}^k + \partial_2 \partial_2 \partial_1 \mathbf{b}^{k-1} = a^{k,0} \partial_1^3 \mathbf{b}^k + 2a^{k,1} \partial_1^2 \mathbf{b}^k + a^{k,2} \partial_1 \mathbf{b}^k. \quad (19)$$

Summing the two instances of (19) and subtracting the two instances of (18) eliminates the mixed derivatives of the left hand side and yields at $(0, 0)$

$$\begin{aligned} 0 &= a^{1,0} \partial_1^3 \mathbf{b}^1 + 2a^{1,1} \partial_1^2 \mathbf{b}^1 + a^{1,2} \partial_1 \mathbf{b}^1 \\ &\quad + a^{3,0} \partial_1^3 \mathbf{b}^3 + 2a^{3,1} \partial_1^2 \mathbf{b}^3 + a^{3,2} \partial_1 \mathbf{b}^3. \end{aligned} \quad (20)$$

Parametric C^2 continuity then implies (17). |||

2.1 Linear α and vertex-localized constructions

The Taylor expansions up to order two of the patches joining at a point are strongly intermeshed by Equation (8). To avoid solving large, global systems, the expansion at a vertex should not depend on the expansions at the neighboring vertices.

Definition 4 (vertex-localized construction) *A surface construction algorithm is G^1 vertex-localized if, at every vertex (with local parameters $(u, v) = (0, 0)$), it sets the second-order Taylor expansion $\partial_1^i \partial_2^j \mathbf{b}^k$, $0 \leq i, j, i + j \leq 2$ independent of the expansions at the neighbor vertices and so that the unbiased G^1 constraints (2) _{$u=0$} and (7) hold.*

A vertex-localized construction enforces the vertex-enclosure constraint (11) at each vertex. We will prefix a statement with ‘**in general**’ to point out that we consider all nonsingular choices of expansions satisfying (11).

Note that a vertex-localized construction can use *a priori* known input, for example the local connectivity and the valence of the neighbors. Nevertheless, the unbiased G^1 constraints imply a *local, unbiased choice of the tangent directions*, namely such that

$$\alpha^k(0) := 2 \cos \frac{2\pi}{n}. \quad (21)$$

(For a proof that logical symmetry implies (21) see e.g. [Pet94, Prop 3].)

Corollary 1 (valence symmetry for $n = 4$ and linear α) *Let $n = 4$ and let n^k denote the valence of the k th neighbor vertex, $k = 1, \dots, n$. Then a local, unbiased choice of the tangent directions and α^k linear are compatible with unbiased G^1 constraints only when the valences of opposite neighbors agree: $n^k = n^{k+2}$.*

Proof The claim follows from Lemma 1 since by the unbiased choice $\alpha^k(0) := 0$ and $\alpha^k(1) := 2 \cos \frac{2\pi}{n^k}$. |||

Corollary 1 is a remarkably strong restriction since vertices of valence $n = 4$ are common. Choosing linear α can therefore be problematic. For example, the construction [HBC08] can therefore not succeed in general.

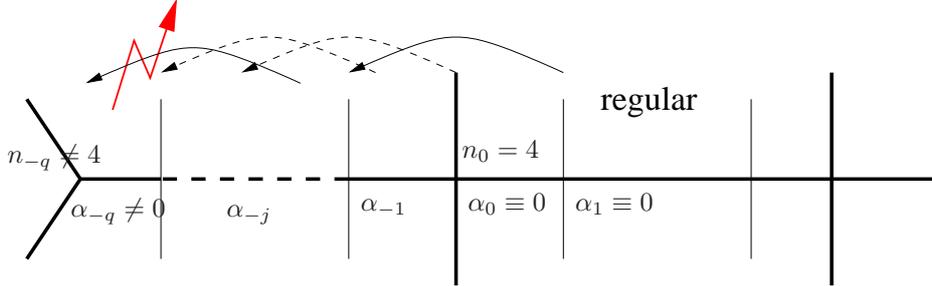


Figure 4: **Propagation** of linear $\alpha_{-j} \equiv 0$ in Lemma 4.

Each scalar function α^k can consist of pieces that correspond to the knot segments of the two generalized splines meeting along the curve. Since, in this context, we only deal with one k at a time, we drop the superscript k and partition the domain of $\alpha := \alpha^k$ by $0 < t_0 < t_1 < \dots < t_q < 1$ into pieces $\alpha_j([0..1]) := \alpha([t_j..t_{j+1}])$. For example, α^3 and α^1 in Fig. 3, are relabeled α_j and α_{j+1} with the parameter adjusted.

Lemma 4 (everywhere piecewise linear α ruled out) *In general, a vertex-localized construction of unbiased G^1 transitions between C^1 generalized spline patches with everywhere at most linear α is not possible.*

Proof Consider a tensor-product regular subgrid where vertex-localized construction implies $\alpha_j \equiv 0$ and where tangents at vertices form an X. Follow a sequence of opposite neighbors across the X configurations (as illustrated in Figure 4) until an irregular vertex \mathbf{p}_{-q} is reached where the tangents do not form an X, e.g. because $n_{-q} \neq 4$ and the construction is vertex-localized. Let \mathbf{p}_0 be the preceding regular vertex (of valence $n_0 = 4$ and whose tangents form an X) Let α_{-1} be the function in the segment from the vertex to its immediate neighbor \mathbf{p}_{-1} (an edge vertex if we have a partition by *edge knots* and $\mathbf{p}_{-1} = \mathbf{p}_{-q}$ otherwise) and $\alpha_0 \equiv 0$ the function for the adjacent segment, in the regular subgrid. By assumption, $\alpha_{-q}(0) \neq 0$.

If α_{-1} is linear then Lemma 1 and $\alpha_0 \equiv 0$ imply $\alpha_{-1} \equiv 0$. If \mathbf{p}_{-1} is an edge vertex, $n_{-1} = 4$ and since the boundary curve is parametrically C^1 , the four tangents at \mathbf{p}_{-1} also form an X. By shifting the focus to the next neighbor and on, we see that X configurations propagate (as illustrated the arrows in Figure 4) when the corresponding α_{-j} are linear. So, whether or not we had edge knots to start with, Lemma 1 then implies that α_{-q} cannot be linear. $\quad |||$

Lastly, we characterize a known source of poor shape of smooth surface constructions due to restricted boundary curves [Pet01]. This limited flexibility is undesirable and constructions that cause it will later be excluded.

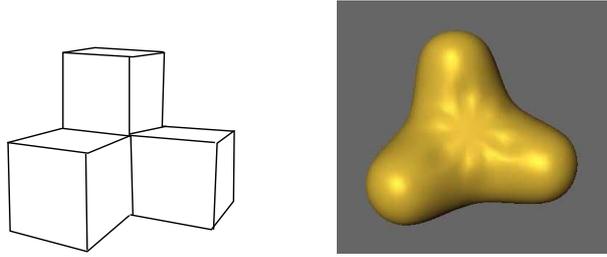


Figure 5: **Shape defect** (star shape) due to embedded straight line segments at a higher order saddle from [URL], *right*.

Lemma 5 (Flatness at saddle points) *Let \mathbf{c} be a curve segment emanating from a higher-order saddle point $\mathbf{p} := \mathbf{c}(0)$. If the derivative \mathbf{c}' of \mathbf{c} factors into a linear, vector-valued polynomial and a scalar factor:*

$$\begin{aligned} \mathbf{c}' &:= \boldsymbol{\ell}\gamma, & (22) \\ \boldsymbol{\ell} : \mathbb{R} &\rightarrow \mathbb{R}^3, \deg(\boldsymbol{\ell}) \leq 1, \quad \gamma : \mathbb{R} \rightarrow \mathbb{R}, \deg(\gamma) \leq 1 \end{aligned}$$

then \mathbf{c} is a planar curve segment. If the saddle is symmetric then \mathbf{c} is a straight line segment.

Proof Let \mathbf{n} be the normal at \mathbf{p} and, without loss of generality, $\gamma(u) := 1 + \gamma_1 u$ for some $\gamma_1 \in \mathbb{R}$. Then $\mathbf{c}'(0) = \boldsymbol{\ell}(0)$, $\mathbf{c}''(0) = \boldsymbol{\ell}'(0) + \boldsymbol{\ell}(0)\gamma_1$ and $\mathbf{c}'''(0) = 2\boldsymbol{\ell}''(0)\gamma_1$. At a higher-order saddle point, the normal curvature is zero, and

therefore $\mathbf{n} \cdot \mathbf{c}''(0) = 0$. This implies $\mathbf{n} \cdot \boldsymbol{\ell}'(0) = 0$ and $\mathbf{n} \cdot \mathbf{c}'''(0) = 0$ establishing planarity. If the saddle is symmetric then $\mathbf{c}'(0)$ and $\mathbf{c}''(0)$ are collinear and so is $\mathbf{c}'''(0) = 2\boldsymbol{\ell}'(0)\gamma_1$. \parallel

A higher-order saddle, such as the monkey saddle of Fig. 5, should have non-zero Gauss curvature apart from the central saddle point. Therefore, we will in the following *disqualify* constructions that force *straight segments* on the boundary for non-flat geometry.

To summarize, we showed that vertex-localized unbiased G^1 constructions with generalized splines are subject to strong restrictions on the reparametrization α (Lemma 1, 2 and 3) or the allowable valence of the vertices (Corollary 1). In the next section, we apply these general restrictions to polynomial splines.

3 Lower bounds for degree bi-3

We now argue that, in general, vertex-localized enforcement of unbiased G^1 constraints with polynomial tensor-product splines of degree bi-3 (bicubic) is possible only if the spline patches have at least two internal double knots per edge.

Since we specialize to polynomials \mathbf{b}^k of degree bi-3, equality in the G^1 constraints implies that α is a rational function, $\alpha =: \frac{\beta}{\gamma}$. In fact, we have a low bound on the degrees of the numerator β and the denominator γ .

Lemma 6 (α degree restricted) *If the two bi-3 patches \mathbf{b}^k and \mathbf{b}^{k-1} satisfy an unbiased G^1 constraint (2) then either*

$$\alpha^k := \frac{\beta}{\gamma} \text{ is rational with} \tag{23}$$

$$(\deg(\beta), \deg(\gamma)) \in \{(2, 1), (2, 0), (1, 1), (1, 0), (0, 1), (0, 0)\}$$

$$\text{and } \partial_1 \mathbf{b}^k(u, 0) = \boldsymbol{\ell}(u)\gamma(u), \deg(\boldsymbol{\ell}) \leq 2 - \deg(\gamma) \tag{24}$$

or the boundary $\mathbf{b}^k(u, 0)$ is forced to have a straight segment.

Proof We may assume that β and γ are relatively coprime. Since the left hand side $\partial_2 \mathbf{b}^k(u, 0) + \partial_1 \mathbf{b}^{k-1}(0, u)$ of the G^1 constraint (2) is polynomial, $\gamma(u)$ must be a (scalar) factor of $\partial_1 \mathbf{b}^k(u, 0) \in \mathbb{R}^3$, the (vector-valued) derivative of the boundary curve. Unless $\mathbf{b}^k(u, 0)$ is a line segment, $0 < \deg(\partial_1 \mathbf{b}^k(u, 0)) \leq 2$. Consequently $\deg(\gamma) \leq 2$ and since $\deg(\gamma) = 2$ implies that $\partial_1 \mathbf{b}^k(u, 0) = \mathbf{v}\gamma$ for a constant vector $\mathbf{v} \in \mathbb{R}^3$, $\deg(\gamma) \leq 1$ must hold to avoid that $\mathbf{b}^k(u, 0)$ is a straight segment. Since $\deg(\partial_2 \mathbf{b}^k(u, 0) + \partial_1 \mathbf{b}^{k-1}(0, u)) \leq 3$, also $\deg(\partial_1 \mathbf{b}^k(u, 0)\beta) \leq 3$ and therefore $\deg(\beta) \leq 2$. \parallel

After scaling numerator and denominator, we may assume that $\gamma(u) := 1 + \gamma_1 u$. A non-linear α then forces a particular boundary curve.

Corollary 2 (Non-linear α restricts boundary curves) *If $(\deg(\beta), \deg(\gamma)) \in \{(2, 1), (2, 0), (1, 1), (0, 1)\}$ then the corresponding degree 3 boundary curve segment is of the form (22).*

Proof The derivative of the curve segment either has a linear factor γ or it is linear because $\deg(\beta) = 2$. |||

Lemma 5 and Corollary 2 together imply that in general, at end points, α must be linear or constant if we require more flexibility than forced straight line segments.

Corollary 3 (Non-linear α at a higher-order saddle) *If $\mathbf{b}^k(u, 0)$ emanates from a symmetric higher-order saddle point and is of degree at most three then α^k in the unbiased G^1 constraints (2) must be linear or constant for $\mathbf{b}^k(u, 0)$ not to be a straight segment.*

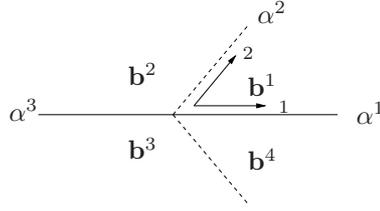


Figure 6: (Figure 3 repeated) Join across an **edge knot** on the boundary (solid) between two splines. The first spline has polynomial pieces \mathbf{b}^1 and \mathbf{b}^2 .

The next lemma shows that at edge knots, neighboring pieces of α constrain one another more than just by (15) and (17).

Lemma 7 (α non-linear at single knot) *Let the segments be arranged as in Figure 6, the edge knot single and the left boundary segment ($\mathbf{b}^3(u, 0)$ shared by the two bi-3 splines) fixed but general (in the sense that the control points cannot be assumed to be in a particular relation). Then α^1 can only be non-linear if*

$$a^{3,0} = 0, a^{3,1} = 0, \text{ and } a^{3,2} = a^{1,2} \neq 0. \quad (25)$$

In particular, α^3 must also be non-linear.

Proof If $\alpha := \alpha^1$ is non-linear then Lemma 6 implies

$$(\deg(\beta), \deg(\gamma)) \in \{(2, 1), (2, 0), (1, 1), (0, 1)\}$$

and therefore $\partial_1 \mathbf{b}^1(u, 0) := \boldsymbol{\ell}(u)\gamma(u)$, a linear vector-valued polynomial times the scalar (possibly constant) factor $\gamma(u) := 1 + \gamma_1 u$. By (14) and (16) and the C^2 constraints for the boundary curve, constraint (17) becomes

$$\text{at } (0, 0), \quad 0 = a^{3,0} \underbrace{(\partial_1^3 \mathbf{b}^3 - 2\gamma_1 \boldsymbol{\ell}'(0))}_{=: \mathbf{v}} + 4a^{3,1} \partial_1^2 \mathbf{b}^3 + (a^{3,2} - a^{1,2}) \mathbf{t}^3. \quad (26)$$

By C^1 continuity $\boldsymbol{\ell}(0)\gamma(0) = \boldsymbol{\ell}(0) = -\mathbf{t}^3$ and hence the C^2 constraint $\partial_1^2 \mathbf{b}^3 = \boldsymbol{\ell}(0)\gamma_1 + \boldsymbol{\ell}'(0) = -\mathbf{t}^3\gamma_1 + \boldsymbol{\ell}'(0)$ implies

$$\boldsymbol{\ell}'(0) = \mathbf{t}^3\gamma_1 + \partial_1^2 \mathbf{b}^3(0, 0). \quad (27)$$

Therefore, at $(0, 0)$, $\mathbf{v} = \partial_1^3 \mathbf{b}^3 - 2\gamma_1(\mathbf{t}^3 \gamma_1 + \partial_1^2 \mathbf{b}^3)$. Since, in general, $\partial_1^3 \mathbf{b}^3(0, 0)$, $\partial_1^2 \mathbf{b}^3(0, 0)$ and \mathbf{t}^3 are linearly independent, the scalar γ_1 can not force $\mathbf{v} = 0$ (recall that \mathbf{b}^3 is fixed), and since \mathbf{v} , $\partial_1^2 \mathbf{b}^3(0, 0)$ and \mathbf{t}^3 are linearly independent, we must have $a^{3,0} = 0$ and $a^{3,1} = 0$ and $a^{1,2} = a^{3,2}$ in order for (26) to hold.

If α^3 is linear then $a^{3,2} = 0$ and since $\alpha''(0) = \left(\frac{\beta}{\gamma}\right)''(0) = \beta''(0)$ when $\alpha(0) = \alpha'(0) = 0$ (note that $\gamma(0) = 1$ and hence $\beta(0) = \beta'(0) = 0$), we have $\alpha^1 \equiv 0$ contradicting the assumption that α^1 is non-linear. $\quad |||$

We now have all the pieces in place to prove the main theorem of smooth surface construction with bi-3 splines.

Theorem 1 (two double edge knots needed) *In general, using splines of degree bi-3 for a vertex-localized unbiased G^1 construction without forced linear boundary segments requires the splines to have at least two internal double knots.*

Proof In general, if the boundary curve has only a single 1-fold knot (hence two C^2 -connected segments) there are not enough degrees of freedom to enforce C^2 continuity of the piecewise curve. If there are two 1-fold knots (three C^2 -connected segments), C^2 continuity uniquely determines all boundary coefficients. If there is one 2-fold knot (two C^1 -connected segments), C^1 continuity uniquely determines all boundary coefficients. However, in these last two cases, (17) is unresolved at the (two, respectively one) edge knots $\{\tau_i\}$ and therefore, in general, these base cases allow for constructing a C^2 boundary curve but not for enforcing (2).

Inserting one additional edge knot that is 1-fold creates one additional boundary curve segment j of degree 3 constrained by four vector-valued constraints: the parametric C^0 , C^1 and C^2 constraints plus (17) or, equivalently, one free spline control point subject to (17). If α_j is linear, its two coefficients are determined via (14) and (16) by those of the neighbor segment, and therefore the free (B-spline) control point must be used to resolve (17). That is, if α_j is linear, we do not gain degrees of freedom that would enable enforcing (17) at the edge knots $\{\tau_i\}$ of the base case.

By Corollary 3, the starting segment's α_0 can be assumed to be linear. Let α_j be non-linear while α_l , $l = 0, \dots, j-1$, $j \geq 1$, are linear. By the reasoning of the previous paragraph all $\mathbf{b}^l(u, 0)$, $l = 0, \dots, j-1$ are determined so that Lemma 7 applies: that is, α_j can only be non-linear if there is at least by one double knot between some segment $\mathbf{b}^{l-1}(u, 0)$ and $\mathbf{b}^l(u, 0)$.

The symmetric argument at the other end implies the claim. $\quad |||$

The proof of Theorem 1 reveals slightly more than its claim: the interior segment with α_j non-linear must be separated by double knots from either end segment. The simplest such construction is then based on three segments with the middle segment bracketed by two double knots, and such that α_0 and α_2 are linear and α_1 quadratic (see Fig. 1, right).

Corollary 4 (lower bound is sharp) *The construction in [FP08] uses the fewest knots when creating a smooth surface that has one bi-3 spline associated with each quad of a general quad mesh and no forced linear segments.*

Proof By covering each quad with a 3×3 arrangement of parametrically C^1 -connected bi-3 patches in Bernstein-Bézier-form, the construction in [FP08] uses exactly two edge knots, both 2-fold. By its choice of quadratic α_1 just for the G^1 constraints across the middle segment and linear α_0 and α_2 for the end segments, it need not have the shape problem characterized by Lemma 5. \square

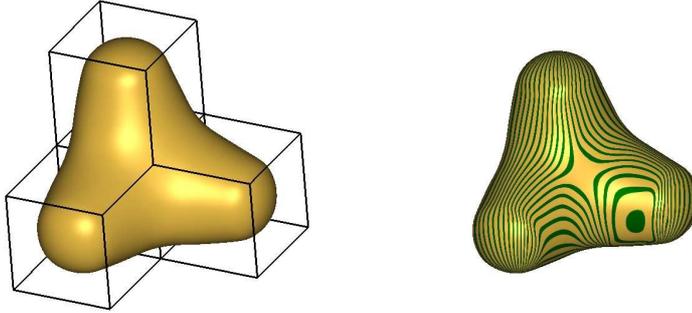


Figure 7: **No Shape defect** (no forced straight line segments) in a higher order saddle (cf. Figure 5).

4 Discussion and Conclusion

Remarkably, the results in Section 2 do not depend on the degree or even the polynomial nature of splines, but assume only sufficiently smooth functions that are piecewise with smooth transitions between the pieces. In particular, the results apply to finite refinement by subdivision which creates parametrically smooth transitions within each generalized spline. The extension to generalized splines mapping to \mathbb{R}^d , $d > 3$ is straightforward.

For bi-3 splines these general constraints imply a lower bound on the number and distribution of knots. The construction in [FP08] shows the lower bound to be tight.

The results extend to constructions based on G^1 transitions of the form $\beta^k(u)\partial_2\mathbf{b}^k(u,0) + \gamma^k(u)\partial_1\mathbf{b}^{k-1}(0,u) = \alpha^k(u)\partial_1\mathbf{b}^k(u,0)$ for which there is a sufficiently rich set of input data that imply $\beta = \gamma$. For example, if $(\alpha^k, \beta^k, \gamma^k)$ reflect the local geometric distribution of the input data, any locally symmetric input yields $\beta = \gamma$ and the results of the paper apply.

The bounds provide a checklist for constructions. Theorem 1 implies for example that there is a subtle error in the proof of the non-trivial construction [HBC08] which uses one double edge knot only: the construction falls foul of Corollary 1. Such a 2×2 split construction can only succeed in special cases. Choosing generic input data, such as two cubes joined at one face to a double-cube initial control net, shows a problem at the splitting points, $\frac{1}{4}$, respectively $\frac{3}{4}$ down the double edges, where $n^1 = n^2 = n^3 = 4$ but $n^4 = 3$. As a second

example, Lemma 4 prevents a vertex-localized solution with all α_j linear. When this lemma is specialized by fixing the degree to be 3, by increasing the patch continuity to C^2 and by choosing $\alpha_{0j} := \frac{q-j}{q}\alpha_{00} + \frac{j}{q}\alpha_{0q}$ then it yields a proof of the claim [SWWL04, Thm 3.1]. (In light of (17), we might adjust the titles of [SWWL04] and [SLW06] since we cannot obtain G^1 surfaces by adding single knots.)

When we restrict connectivity, i.e. drop the assumption made at the outset that the construction applies to general input and uses one tensor-product spline per quad, then constructions with fewer edge knots are possible. For restricted connectivity, it is well known that if all valences are odd or tangents form an X, then vertex-enclosure does not impose constraints and simple Bézier constructions are possible (e.g. [vW86, Pet91b, GZ94]). If $n^0 = n^1$ always holds, say when smoothing a cube, then we can choose linear α^1 and α^3 with $a^{1,1} = a^{3,1}$ and $a^{1,0} = 0$ to enforce (15). That is, a construction with one double edge knot is possible. Such a construction, covering a quad by 2×2 bi-3 patches, is proposed in [HBC08]. A similar but dual, spline-like construction appears in [ZT95]. Global constructions, singular parameterizations [Pet91a, NP94, Rei98], control of the valence, for example by splitting patches [PBP02, 9.11], [Pet91b, Pet95b], can allow for simpler constructions. Reif's G^1 construction [Rei95] uses multiple patches per quad, not following the tensor-product spline paradigm but with some biased transitions. Remarkably, the patches are of degree bi-2. But this implies undesirable boundary curves of type (5).

If we allow higher degree, then general constructions of smooth surfaces with one patch per quad are shown possible for degree bi-5, for example [MYP08]. For degree bi-4, a single knot (a 2x2-split) must be introduced (see e.g. [Pet95a]).

The case of several G^1 -connected patches per quad still awaits full investigation, as does the case of rational bi-3 patches and the generalization of the problem to unbiased G^k transitions for $k > 1$.

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