A Geometric Criterion for Smooth Interpolation of Curve Networks

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Abstract

A key problem when interpolating a network of curves occurs at vertices: an *algebraic* condition called the vertex enclosure constraint must hold wherever an even number of curves meet. This paper recasts the constraint in terms of the local geometry of the curve network. This allows formulating a new *geometric* constraint, related to Euler's Theorem on local curvature, that implies the vertex enclosure constraint and is equivalent to it where four curve segments meet without forming an X.

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1 Introduction

A common approach to surface modeling is to create a network of curves and then determine a regular surface that interpolates the network (see Figure 2,*left*). In this approach, the curve segments meeting at a vertex are constructed to be regular and smooth and their tangents are placed into the same plane to indicate that a tangent continuous surface is sought. The main challenge occurs at each vertex $\mathbf{p}$: we need to find a second order expansion of the n surface pieces $\mathbf{x}_i$ that interpolate the second order expansions of the curves $\mathbf{c}_j$ meeting at $\mathbf{p}$ (Figure 2,*right*).

It is known, for example from [Bez77; vW86; Wat88; Sar87; Sar89; DS91; Ren91], that interpolating the second order expansions is always and uniquely possible if the number of curves meeting is odd — but that, when the number of curves is even, an additional algebraic constraint must hold for the *normal components* of the curve expansions. This is the vertex enclosure constraint [Pet91b].

This paper gives a new formulation of the vertex enclosure constraint in terms of the *local geometry of the curve network*. Such a formulation is helpful as a criterion for the layout of admissible curve networks and is complementary to work on improving shape such as [Pot92]. We do not weigh down the paper with specific strategies for adjusting curves and constructing the actual surfaces

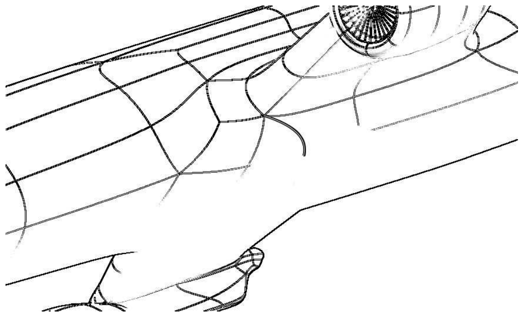

Figure 1: *Network of curve segments.*

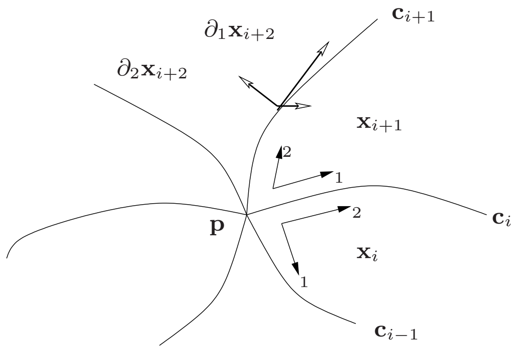

Figure 2: This paper focuses on local network interpolation (see Definition 1, cf. [HL 93, Fig 7.12]): curves $\mathbf{c}_j$, $j \in \mathbb{Z}_n$, meeting at a point $\mathbf{p}$ are given and pairwise interpolating patches $\mathbf{x}_j$ are sought. The arrow labels $_1$ and $_2$ indicate the domain parameters associated with the boundary curves of the patches, e.g. $\partial_1\mathbf{x}_{i+2} = \partial_2\mathbf{x}_{i+1}$.

since there are a myriad of choices, typically specific to the design context and preferred representation of the interpolating surface, but focus on the fundamental geometric insight.

2 Local curve interpolation: overview and background

As illustrated in Figure 2, we consider n curves $\mathbf{c}_i : \mathbb{R} \to \mathbb{R}^3$ that start at a point $\mathbf{p} \in \mathbb{R}^3$ and have tangents $\mathbf{t}_i \in \mathbb{R}^3$ at $\mathbf{p}$ that admit a common normal $\mathbf{n} \in \mathbb{R}^3$. Let $\mathbf{x}_i : \mathbb{R}^2 \to \mathbb{R}^3$, $i \in \mathbb{Z}_n$ be n patches filling-in between the curves. If each patch is C^2 at $\mathbf{p}$, as would be the case for piecewise polynomial patches such as C^2 spline patches, then, in order for an *even* number n of patches to pairwise meet along the curves with tangent continuity (and, in particular, at $\mathbf{p}$), the curves have to satisfy one scalar constraint. This scalar constraint, the first-order *vertex enclosure constraint* (6), links the *normal component* of the second derivatives of the curves.

For an odd number of patches, no vertex enclosure constraint exists

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(Theorem 1). If $n = 4$ and the curve tangents $\mathbf{t}_j$, $j = 1, 2, 3, 4$ form an X, i.e., they are pairwise collinear $\mathbf{t}_i \times \mathbf{t}_{i+2} = 0$ for $i = 1, 2$, then the constraint holds automatically (Lemma 3). For example, this holds at an internal knot of a smooth tensor-product spline where four polynomial pieces meet. Also, if we use patches $\mathbf{x}_i$ that are not C^2 at $\mathbf{p}$, no vertex enclosure constraint applies. The three known classes of techniques that avoid the vertex enclosure constraint are: singular patch constructions, e.g. [BR97; NP94; Pet91a], rational base-point constructions, e.g. [Gre74; CK83], and split-patch constructions [Pip87; Pet95]. Here the patches are either not C^2 or not regular. Since we are interested in the standard case where the patches are regular and C^2, we will in the following assume that the patches satisfy, locally at the point $\mathbf{p}$, the 'compatibility condition' $\partial_1\partial_2\mathbf{x} = \partial_2\partial_1\mathbf{x}$. In particular, tensor-product spline patches, even with multiple knots such as C^1 bicubics, satisfy the compatibility constraint since, at their corners where patches abut, they consist of a single polynomial piece.

For an even number of patches, the compatibility condition together with the first-order continuity contraints result in a rank deficient circulant system that is only solvable if we restrict its right hand side [Bez77; Wat88; vW86; Sar87; Sar89; DS91; Ren91]. Du and Schmitt [DS91] gave a sufficient condition for this vector-valued constraint to hold but did not connect this to the geometric characteristics of the curve network. An important insight [Pet91b] is that, for the components in the tangent plane, we have plenty of freedom to satisfy the system. The essential part of the constraint is the normal component. We therefore separate the components in the normal direction from those in the tangent plane and focus on the fact that the system with the normal component as right hand side can only be solved if an alternating sum of the right hand side vanishes. In Section 3, we derive equation (6) that re-interprets this algebraic constraint geometrically in terms of angles and normal curvatures.

In [Pet91b], it was shown that if there exists a symmetric (embedded Weingarten) matrix $\mathbf{W} \in \mathbb{R}^{3\times3}$ such that the boundary curves $\mathbf{c}_i$ all satisfy

$$\mathbf{n} \cdot \partial_j\partial_k\mathbf{x}_i = (\mathbf{t}_j)^{\mathrm{t}}\mathbf{W}\mathbf{t}_k \tag{1}$$

then the vertex enclosure constraint holds. The relation was called *compatibility with a second fundamental form*. In Section 4, we give an equivalent, alternative sufficient condition that implies that the vertex enclosure constraint holds. This relation is similar to Euler's theorem — which states that if $\kappa^{\mathbf{n}}$ is the normal curvature of a curve emanating from a point on a C^2 surface then there exist scalars κ_1, κ_2 and an angle ϕ with respect to a fixed direction $\mathbf{d}$ in the tangent plane so that $\kappa^{\mathbf{n}} = \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$.

3 The C^1 vertex enclosure constraint

We now derive the vertex enclosure constraint and show it to be equivalent to a relation involving only the angles between the tangents and the normal curvature. We can focus on the normal component since the tangential components will not restrict the curves (see Theorem 1 below). We define the overall task.

Definition 1 (Smooth Network Interpolation)**.** *Let*

$$\mathbf{c}_i : \mathbb{R} \to \mathbb{R}^3, t \mapsto \mathbf{c}_i(t), \qquad i \in \mathbb{Z}_n \tag{2}$$

be a sequence of regular, C^2 curves in $\mathbb{R}^3$ that meet at a common point $\mathbf{p}$ in a plane with oriented normal $\mathbf{n}$ and at angles ψ_i less than π:

$$\mathbf{c}_i(0) = \mathbf{p}, \mathbf{c}_i'(0) =: \mathbf{t}_i \perp \mathbf{n}, \quad 0 < \psi_i := \angle(\mathbf{t}_i, \mathbf{t}_{i+1}) < \pi. \tag{3}$$

Define a smooth network interpolation *of $\{\mathbf{c}_i\}$ to be a sequence of patches*

$$\mathbf{x}_i : \mathbb{R}^2 \to \mathbb{R}^3, (u, v) \mapsto \mathbf{x}_i(u, v), \qquad i \in \mathbb{Z}_n \tag{4}$$

that are regular and C^2 at $\mathbf{p}$, that interpolate the curve network according to $\mathbf{x}_i(t, 0) = \mathbf{c}_{i-1}(t)$ and $\mathbf{x}_i(0, t) = \mathbf{c}_i(t)$, and that connect pairwise so that the G^1 constraints, i.e., C^1 continuity after reparameterization (see e.g. [PBP02] or [Pet02]), hold for scalar functions α_i, β_i and γ_i:

$$\alpha_i(t)\partial_1\mathbf{x}_i(0, t) + \beta_i(t)\partial_2\mathbf{x}_i(0, t) + \gamma_i(t)\partial_2\mathbf{x}_{i+1}(t, 0) = 0. \tag{5}$$

Smooth Network Interpolation restricted to the neighborhood of $\mathbf{p}$ is called local network interpolation.

Note that the angle ψ_i corresponds to patch $\mathbf{x}_{i+1}$. To simplify computations, we assume in the following, for notational convenience only, that the curves are arclength-parameterized up to second order at $t = 0$. In particular, $\mathbf{t}_i$ is a unit vector. Our results will nevertheless also apply to polynomial patches in their original form. The normal curvature of $\mathbf{c}_i$ at $\mathbf{p}$,

$$\kappa_i^{\mathbf{n}} := \mathbf{c}_i''(0) \cdot \mathbf{n},$$

will play a central role in the following.

Theorem 1 (vertex enclosure constraint)**.** *If n is odd then local network interpolation is always possible. If n is even then local network interpolation is possible if and only if the vertex enclosure constraint* (6) *holds*

$$0 = \sum_{i=1}^{n} (-1)^i (\cot\psi_{i-1} + \cot\psi_i)\kappa_i^{\mathbf{n}}. \tag{6}$$

Alternatively, when n is even, we can write (6) as

$$\begin{aligned} 0 &= \sum_{i=1}^{n} (-1)^i \kappa_i^{\mathbf{n}} (\cot\psi_{i-1} + \cot\psi_i) \qquad (7) \\ &= \sum_{i=1}^{n} (-1)^i (\kappa_i^{\mathbf{n}} - \kappa_{i+1}^{\mathbf{n}}) \cot\psi_i. \end{aligned}$$

Proof. Abbreviating $a_i := \alpha_i(0), b_i := \beta_i(0), c_i := \gamma_i(0)$, (5) implies

$$a_i\mathbf{t}_{i-1} + b_i\mathbf{t}_i + c_i\mathbf{t}_{i+1} = 0. \tag{8}$$

Taking the cross product with $\mathbf{t}_i$ and then the scalar product of the result with $\mathbf{n}$ yields

$$a_i \det[\mathbf{t}_{i-1}, \mathbf{t}_i, \mathbf{n}] + c_i \det[\mathbf{t}_{i+1}, \mathbf{t}_i, \mathbf{n}] = 0.$$

Since $0 < \psi_i < \pi$, $\det[\mathbf{t}_{i-1}, \mathbf{t}_i, \mathbf{n}] > 0$ and since the surface pieces are regular at $(0, 0)$, $c_i \neq 0 \neq a_i$, and hence

$$a_i = c_i \frac{\det[\mathbf{t}_i, \mathbf{t}_{i+1}, \mathbf{n}]}{\det[\mathbf{t}_{i-1}, \mathbf{t}_i, \mathbf{n}]} = c_i \frac{\sin\psi_i}{\sin\psi_{i-1}}. \tag{9}$$

Similarly,

$$b_i = -c_i \frac{\det[\mathbf{t}_{i-1}, \mathbf{t}_{i+1}, \mathbf{n}]}{\det[\mathbf{t}_{i-1}, \mathbf{t}_i, \mathbf{n}]} = -c_i \frac{\sin(\psi_{i-1} + \psi_i)}{\sin\psi_{i-1}}. \tag{10}$$

Differentiating both sides of (5) along the common boundary yields the system of n equations, $i \in \mathbb{Z}_n$

$$a_i\mathbf{w}_i + b_i\mathbf{k}_i + c_i\mathbf{w}_{i+1} + a_i'\mathbf{t}_{i-1} + b_i'\mathbf{t}_i + c_i'\mathbf{t}_{i+1} = 0, \tag{11}$$

where we abbreviated the *corner twist vector* of the patch $\mathbf{x}_i$ at $\mathbf{p}$ as

$$\mathbf{w}_i := \partial_1\partial_2\mathbf{x}_i(0, 0), \tag{12}$$

the *curvature vector* of $\mathbf{c}_i$ at $\mathbf{p}$ as

$$\mathbf{k}_i := \partial_2^2\mathbf{x}_i(0, 0) = \mathbf{c}_i''(0) \tag{13}$$

and

$$a_i' := \alpha_i'(0),\ b_i' := \beta_i'(0),\ c_i' := \gamma_i'(0). \tag{14}$$

As we form the scalar product of each side of (11) with $\mathbf{n}$, we obtain

$$a_i \mathbf{w}_i \cdot \mathbf{n} + b_i \mathbf{k}_i \cdot \mathbf{n} + c_i \mathbf{w}_{i+1} \cdot \mathbf{n} = 0 \qquad (i \in \mathbb{Z}_n). \tag{15}$$

Setting $q_i := \mathbf{w}_i \cdot \mathbf{n}$, (15) simplifies to

$$a_i\ q_i + b_i \kappa_i^{\mathbf{n}} + c_i\ q_{i+1} = 0, \quad i \in \mathbb{Z}_n. \tag{16}$$

On substituting a_i and b_i using (9) and (10), the equations (16) become

$$\frac{\sin\psi_i}{\sin\psi_{i-1}} q_i - \frac{\sin(\psi_{i-1}+\psi_i)}{\sin\psi_{i-1}} \kappa_i^{\mathbf{n}} + q_{i+1} = 0 \tag{17}$$

and therefore

$$\frac{q_i}{\sin\psi_{i-1}} + \frac{q_{i+1}}{\sin\psi_i} = (\cot\psi_{i-1} + \cot\psi_i)\kappa_i^{\mathbf{n}}. \tag{18}$$

With the introduction of the variables

$$\tilde{q}_i := \frac{q_i}{\sin\psi_{i-1}}$$

and

$$\tilde{\kappa}_i := (\cot\psi_{i-1} + \cot\psi_i)\kappa_i^{\mathbf{n}}$$

our equation takes the simple form

$$\tilde{q}_i + \tilde{q}_{i+1} = \tilde{\kappa}_i \quad i \in \mathbb{Z}_n. \tag{19}$$

Alternately subtracting and adding Equations 19, we see

$$\tilde{q}_1 = (-1)^n \tilde{q}_1 + \sum_{i=1}^{n} (-1)^{i-1} \tilde{\kappa}_i. \tag{20}$$

If n is even then the system of equations (19) is singular and by (20) there is a solution only if

$$\sum_{i=1}^{n} (-1)^i \tilde{\kappa}_i = 0. \tag{21}$$

This establishes the necessity of (6).

We now show sufficiency: if the boundary curves $\{\mathbf{c}_i\}_{i\in\mathbb{Z}_n}$ either satisfy (6) or if n is odd then the system (15) can be solved for $q_i := \mathbf{w}_i \cdot \mathbf{n}$ and admits local network interpolation. If n is *even* and (6) holds then the solution is not unique. For example, [Pet00], gives a solution. If n is *odd* then, by (20),

$$\tilde{q}_1 = -\frac{1}{2} \sum_{i=1}^{n} (-1)^i \tilde{\kappa}_i \tag{22}$$

and we can backsolve $q_j = \tilde{\kappa}_{j-1} - q_{j-1}$ for $j = 2, \ldots, n$. In either case, since $\{\mathbf{t}_{i-1}, \mathbf{t}_i, \mathbf{t}_{i+1}\}$ spans the tangent space, (11) is solvable for some, possibly non-unique set, $\{(\mathbf{w}_i, a_i', b_i', c_i')\}$. Then

$$\begin{aligned} \mathbf{x}_i(u,v) &:= \mathbf{c}_{i-1}(u) + \mathbf{c}_i(v) - \mathbf{p} \\ &\quad + uv\big(\mathbf{w}_i + u\mathbf{l}_i(u) + v\mathbf{r}_i(v) + uv\mathbf{m}_i(u,v)\big) \end{aligned}$$

is well-defined and is the unique interpolant of $\mathbf{c}_{i-1}$, $\mathbf{c}_i$, and $\mathbf{w}_i$ up to a choice of univariate functions $\mathbf{l}_i$ and $\mathbf{r}_i$, and some bivariate function $\mathbf{m}_i$. We obtain a local network interpolation, for example for the choice $\alpha_i(t) := a_i + a_i' t$, $\beta_i(t) := b_i + b_i' t$, $\gamma_i(t) := c_i + c_i' t$, and $\hat{\mathbf{c}}_i(t) := \mathbf{c}_i(t) - \mathbf{p} - \mathbf{t}_i t - \mathbf{k}_i t^2/2$ by setting

$$\mathbf{r}_i(t) := \mathbf{l}_i(t) := -\frac{a_i'\mathbf{w}_i + b_i'\mathbf{k}_i + c_i'\mathbf{w}_{i+1} + \beta_i(t)\hat{\mathbf{c}}_i'(t)}{\alpha_i(t) + \gamma_i(t)}. \tag{23}$$

The denominator is nonzero in a neighborhood of 0 since $\alpha_i(0) + \gamma_i(0) \neq 0$ by (3) and (9). Combining like terms of t, the left side of (5) becomes

$$\begin{aligned} &\alpha_i(t)\left(\mathbf{t}_{i-1} + t(\mathbf{w}_i + t\mathbf{r}_i(t))\right) + \beta_i(t)(\mathbf{t}_i + \mathbf{k}_i t + \hat{\mathbf{c}}_i')(t) \\ &\quad + \gamma_i(t)\left(\mathbf{t}_{i+1} + t(\mathbf{w}_{i+1} + t\mathbf{l}_{i+1}(t))\right) \\ &= (a_i\mathbf{t}_{i-1} + b_i\mathbf{t}_i + c_i\mathbf{t}_{i+1}) \\ &\quad + t(a_i'\mathbf{t}_{i-1} + b_i'\mathbf{t}_i + c_i'\mathbf{t}_{i+1} + a_i\mathbf{w}_i + b_i\mathbf{k}_i + c_i\mathbf{w}_{i+1}) \\ &\quad + t^2(a_i'\mathbf{w}_i + b_i'\mathbf{k}_i + c_i'\mathbf{w}_{i+1} \\ &\quad + \alpha_i(t)\mathbf{r}_i(t) + \beta_i(t)\hat{\mathbf{c}}_i'(t) + \gamma_i(t)\mathbf{l}_{i+1}(t)) \\ &=^{(8),(11),(23)} 0 \end{aligned} \tag{24}$$

Therefore (5) holds. □

Next, we will derive a geometric constraint that implies the geometric interpretation (6) of the vertex enclosure constraint.

4 Vertex Enclosure and Euler's Theorem

For a point on a C^2 surface, Euler's Theorem expresses normal curvature in any tangent direction in terms of the principal curvatures: (see e.g. [dC76, page 145])

$$\kappa^{\mathbf{n}} = \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi \tag{25}$$

where ϕ is the angle between $\mathbf{t}$ and the principial direction of κ_1. We will aim to enforce a similar constraint for the curve network.

Definition 2 (Euler curvature constraint)**.** *Let n be even and $\{\phi_i\}_{i\in\mathbb{Z}_n}$ such that $\phi_{i+1} - \phi_i = \psi_i$, and $\mathbf{c}_i$, $i \in \mathbb{Z}_n$ be curves whose tangents form angles ϕ_i from some fixed direction $\mathbf{d}$. Then the* Euler curvature constraint *holds for curves $\mathbf{c}_i$, with normal curvatures $\kappa_i^{\mathbf{n}} \in \mathbb{R}$, $i \in \mathbb{Z}_n$ if there exist constants $\kappa_1, \kappa_2 \in \mathbb{R}$ such that*

$$\kappa_i^{\mathbf{n}} = \kappa_1 \cos^2\phi_i + \kappa_2 \sin^2\phi_i, \quad i \in \mathbb{Z}_n. \tag{26}$$

In the following, we will see that this newly defined geometric Euler constraint and the vertex enclosure constraint are closely linked.

Theorem 2 (The Euler curvature constraint implies the vertex enclosure constraint)**.** *If the Euler constraint* (26) *holds for curves $\mathbf{c}_i$, $i \in \mathbb{Z}_n$, then the vertex enclosure constraint* (6) *holds.*

Proof. Below we use (26) and the trigonometric identities

$$\begin{aligned} \kappa_i^{\mathbf{n}} - \kappa_{i+1}^{\mathbf{n}} &=^{(26)} (\kappa_2 - \kappa_1)\left(\sin^2\phi_i - \sin^2\phi_{i+1}\right) \\ &= (\kappa_1 - \kappa_2)\sin(\phi_i + \phi_{i+1})\sin(\phi_{i+1} - \phi_i) \quad (27) \\ &= (\kappa_1 - \kappa_2)\sin(\phi_i + \phi_{i+1})\sin\psi_i, \end{aligned}$$

$$2\sin(\sigma+\tau)\cos(\sigma-\tau) = \sin 2\sigma + \sin 2\tau, \tag{28}$$

and, in the last equality, that n is even. Then the right hand side of (6),

$$\sum_{i=1}^{n} (-1)^i (\cot\psi_{i-1} + \cot\psi_i)\kappa_i^{\mathbf{n}} = \sum_{i=1}^{n} (-1)^i \frac{\cos\psi_i}{\sin\psi_i} \left(\kappa_i^{\mathbf{n}} - \kappa_{i+1}^{\mathbf{n}}\right) \tag{29}$$

$$= (\kappa_1 - \kappa_2) \sum_{i=1}^{n} (-1)^i \cos(\phi_{i+1} - \phi_i)\sin(\phi_i + \phi_{i+1}) \tag{30}$$

$$= \frac{\kappa_1 - \kappa_2}{2} \sum_{i=1}^{n} (-1)^i (\sin 2\phi_{i+1} + \sin 2\phi_i) = 0 \tag{31}$$

as claimed. □

Conversely, however, (6) does not, in general, imply (26). This is shown by next two lemmas. The first appeared in similar form in [PW92; Her96].

Lemma 1 (Determining κ_1, κ_2 and $\mathbf{d}$ from three curves)**.** *Let* $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ *be tangent vectors:* $\mathbf{n}\cdot\mathbf{t}_i = 0$*. If no pair of* $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ *is parallel then* κ_1*,* κ_2 *and* $\mathbf{d}$ *of* (26) *can be determined from the curves* $\mathbf{c}_i$*,* $i = 1, 2, 3$.

Proof. Choosing without loss of generality the coordinates so that

$$\mathbf{n} := \begin{bmatrix}0\\0\\1\end{bmatrix}, \mathbf{t}_1 := \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{t}_2 := \begin{bmatrix}x_2\\y_2\\0\end{bmatrix}, \mathbf{t}_3 := \begin{bmatrix}x_3\\y_3\\0\end{bmatrix},$$

$$\mathbf{W} := \begin{bmatrix}W & 0\\0 & 1\end{bmatrix} := \begin{bmatrix}w_1 & w_2/2 & 0\\w_2/2 & w_3 & 0\\0 & 0 & 1\end{bmatrix},$$

the constraints (1) for $i = 1, 2, 3$,

$$\kappa_i^{\mathbf{n}}\|\mathbf{t}_i\|^2 = (\mathbf{t}_i)^{\mathrm{t}}\mathbf{W}\mathbf{t}_i = (\mathbf{t}_i(1))^2 w_1 + (\mathbf{t}_i(1)\mathbf{t}_i(2))w_2 + (\mathbf{t}_i(2))^2 w_3$$

yield the system

$$T\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} := \begin{bmatrix}1 & 0 & 0\\x_2^2 & x_2y_2 & y_2^2\\x_3^2 & x_3y_3 & y_3^2\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}\kappa_1^{\mathbf{n}}\|\mathbf{t}_1\|^2\\\kappa_2^{\mathbf{n}}\|\mathbf{t}_2\|^2\\\kappa_3^{\mathbf{n}}\|\mathbf{t}_3\|^2\end{bmatrix}$$

to be solved for w_1, w_2, w_3. The 3×3 matrix T is invertible since

$$\det T = y_2 y_3 \det\begin{bmatrix}x_2 & y_2\\x_3 & y_3\end{bmatrix}$$

and by assumption $y_2 \neq 0 \neq y_3$ and $\mathbf{t}_2$ and $\mathbf{t}_3$ are not collinear. We can now choose κ_1 and κ_2 as the eigenvalues of the 2×2 submatrix W and the direction $\mathbf{d}$ as the eigenvector of κ_1. This uniquely defines an embedded Weingarten map. Therefore Euler's Theorem holds and this implies (26). □

Given this linear relationship between data and curvature, we cannot expect that the vertex enclosure constraint (6) implies the Euler constraint (26). For example, when the number of curves is $n = 6$ then we can choose the curvature of five of the curves to not satisfy (26). But, due to the linear dependence of the curvatures in (6), we can choose the curvature of the sixth curve so that (6) holds. The following example makes this concrete.

Lemma 2. *The vertex enclosure constraint* (6) *is weaker than the Euler constraint* (26)*.*

Proof. Choose

$$\psi_1 = \psi_2 = \psi_3 = \frac{2\pi}{12}, \psi_4 = \psi_5 = \psi_6 = \frac{2\pi}{4},$$
$$\kappa_1^{\mathbf{n}} = \kappa_2^{\mathbf{n}} = \kappa_3^{\mathbf{n}} = \kappa_4^{\mathbf{n}} = 0, \kappa_5^{\mathbf{n}} = \kappa_6^{\mathbf{n}} = 1.$$

Then by Lemma 1, since the directions defined by ψ_1, ψ_2, and ψ_3 are not pairwise dependent, $\kappa_1^{\mathbf{n}} = \kappa_2^{\mathbf{n}} = \kappa_3^{\mathbf{n}} = 0$ implies that the right hand side of (26) is zero and this is not consistent with $\kappa_5^{\mathbf{n}} = \kappa_6^{\mathbf{n}} = 1$. However, since $\sin(\psi_4+\psi_5) = \sin(\psi_5+\psi_6) = 0$, (6) holds. □

5 Four curve segments meeting

We now focus on the case $n = 4$. We show that (6) and (26) are equivalent, unless the tangents form an X.

We start with the exception, giving an example where (6) holds but not (26). We note that if the four tangents form an X then Lemma 1 does not apply.

Lemma 3. *If four curve segments meet forming an X then* (6) *holds automatically and does not imply* (26)*.*

Proof. Equations (6) hold without restriction on the $\kappa_i^{\mathbf{n}}$ since $\sin(\psi_{i-1} + \psi_i) = 0$. On the other hand, $\kappa_1^{\mathbf{n}} = \kappa_1 + (\kappa_2 - \kappa_1)\sin^2\phi_1$ and $\kappa_3^{\mathbf{n}} = \kappa_1 + (\kappa_2 - \kappa_1)\sin^2(\phi_1 + \pi) = \kappa_1^{\mathbf{n}}$ so that (26) implies

$$\kappa_1^{\mathbf{n}} - \kappa_3^{\mathbf{n}} = 0 = \kappa_2^{\mathbf{n}} - \kappa_4^{\mathbf{n}}. \tag{32}$$

That is, each pair of curves should have equal normal curvatures. For an example where (6) holds but not (26), we choose the curves so that $\kappa_1^{\mathbf{n}} = \kappa_2^{\mathbf{n}} = \kappa_3^{\mathbf{n}} = 0$ but $\kappa_4^{\mathbf{n}} = 1$. □

Generically, however, contrary to [Pet91b, Claim 3.3], Equations (6) are equivalent to (26).

Lemma 4 (Equivalence of vertex enclosure and Euler curvature constraint for $n = 4$)**.** *If the vertex enclosure constraint* (6) *holds for* $n = 4$ *and the tangents* $\mathbf{t}_i$ *are not pairwise parallel then*

$$\kappa_i^{\mathbf{n}} = \kappa_1\cos^2\phi_i + \kappa_2\sin^2\phi_i \tag{33}$$

for some choice of κ_1*,* κ_2 *and angles* ϕ_i *measured from some fixed direction* $\mathbf{d}$*.*

Proof. Without loss of generality, let $\mathbf{d}$ be the leg from which the $\{\phi_i\}$ are measured and assume that $\mathbf{t}_2$ and $\mathbf{t}_4$ are not parallel so that $\sin(\psi_1 + \psi_4) \neq 0$. By Lemma 1, we can determine the direction and the scalars κ_1 and κ_2 so that (33) holds for $i \in \{2, 3, 4\}$. Let $r := \kappa_1\cos^2\phi_1 + \kappa_2\sin^2\phi_1$ and $\kappa_1^{\mathbf{n}}$ be the normal curvature of $\mathbf{c}_1$ that we want to show equal to r. If we replace $\kappa_1^{\mathbf{n}}$ with r (6) is satisfied (by Theorem 2). Thus

$$\begin{aligned} 0 &= \sum_{i=1}^{n}(-1)^i(\cot\psi_{i-1} + \cot\psi_i)\kappa_i^{\mathbf{n}} \\ &= (\cot\psi_4 + \cot\psi_1)(\kappa_1^{\mathbf{n}} - r)) \\ &= \frac{\sin(\psi_1 + \psi_4)}{\sin\psi_4\sin\psi_1}(r - \kappa_1^{\mathbf{n}}). \end{aligned}$$

The denominator in the last expression is non-zero since, by the initial assumption of the paper, $0 < \psi_i < \pi$. Since $\sin(\psi_1+\psi_4) \neq 0$, the claim follows. □

6 Conclusion and further challenges

The main result of this paper is the geometric formulation (6) of the vertex enclosure constraint. This allows in particular to derive the Euler curvature constraint, (26). We showed that (26) implies (6) but that the two conditions are generally not equivalent unless $n = 4$ curve segments meet without forming an X. Note that the results do not depend on a particular, say spline, representation. For example, the curves $\mathbf{c}_i$ can be procedurally-defined intersection curves.

Since cusps occur in industrial practice, for example for some blending applications, it would be good to extend the theory to the case when some angle between curves is $\psi_i = 0$. A more ambitious challenge is to find the vertex enclosure constraint for the curvature continuous case or show that no such constraint is needed.

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