International Journal of Computational Geometry & Applications

Reconciling conflicting combinatorial preprocessors for geometric constraint systems

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Received (received date)
Revised (revised date)
Communicated by (Name)

Polynomial equation systems arising from real applications often have associated combinatorial information, expressible as graphs and underlying matroids. To simplify the system and improve its numerical robustness before attempting to solve it with numeric-algebraic techniques, solvers can employ graph algorithms to extract substructures satisfying or optimizing various combinatorial properties. When there are underlying matroids, these algorithms can be greedy and efficient. In practice, correct and effective merging of the outputs of different graph algorithms to *simultaneously* satisfy their goals is a key challenge.

This paper merges and improves two highly effective but separate graph-based algorithms that preprocess systems for resolving the relative position and orientation of a collection of incident rigid bodies. Such collections naturally arise in many situations, for example in the recombination of decomposed large geometric constraint systems. Each algorithm selects a subset of incidences, one to optimize algebraic complexity of a parametrized system, the other to obtain a well-formed system that is robust against numerical errors. Both algorithms are greedy and can be proven correct by revealing underlying matroids. The challenge is that the output of the first algorithm is not guaranteed to be extensible to a well-formed system, while the output of the second may not have optimal algebraic complexity. Here we show how to reconcile the two algorithms by revealing well-behaved maps between the associated matroids.

Keywords: Combinatorial preprocessing of algebraic systems; Graph-based optimization of Algebraic Complexity; 3D geometric constraint systems; Solving Polynomial Systems.

1. Introduction

Graph-based preprocessing algorithms have a long history for example in the numerical treatment of sparse linear systems [1, 5]. Similarly, graphs play a key role in recursively decomposing industrial-size non-linear geometric constraint systems [7].

[*]Corresponding author, Work supported in part by NSF Grant CCR 99-02025, NSF Grant EIA 00-96104 and a grant from SolidWorks
[†]Work supported in part by NSF Grant CCR DMI-0400214 and CCF-0430891

More recently, the 'overlap graph' and the 'seam graph' and underlying matroids have been leveraged for efficient preprocessing of polynomial equation systems arising from incident collections of rigid bodies.

Given a collection of incident rigid bodies, a *seam graph* is used to select a *well-formed* subset of incidences. The resulting well-formed system of incidences ensures a correspondence between each solution of the original system and a solution to the numerically perturbed, well-formed system [19].

The *overlap graph* is used to choose and order the elimination of incidences, i.e., to select a sequence of parametrizations that minimizes the number of variables and the algebraic degree of the parametrized core system (*minimizing algebraic complexity*) [20].

This paper makes the following contributions, illustrated in Figure 7, p. 15.

- A simple example to show that well-formed systems of incidences generally do not optimize algebraic complexity (Figure 9, Section 6) . The Well-formed Incidences Algorithm [19] , in particular, does not optimize algebraic complexity.
- A second example to show that choosing well-formed systems is nontrivial even for small systems (Figure 3); and the Optimal Incidence Tree Algorithm [20] does not guarantee well-formedness, i.e. redundant constraints may be selected.
- A new algorithm that combines the key elements of the Optimal Incidence Tree Algorithm with the Well-formed Incidences Algorithm to guarantee both optimal algebraic complexity and well-formedness. This greedy and efficient algorithm is the result of a careful construction and proof of an independence-preserving map between the cycle matroid associated with the overlap graph and two other matroids that identifying well-formed systems of incidences.

We note again that the combinatorial or graph algorithm *pre-processes* is to be followed by a -now simpler- algebraic or numeric solving stage. That is the algorithm assumes algebraically generic input and is not tied to a specific solver. After a short overview of the history and application area of the two algorithms, Section 2 introduces systems of incidences between collections of maximal rigid bodies with the help of a simple Example 1. Example 1 illustrates the concepts in all sections. Section 3 summarizes the complexity optimization algorithm of [20]. Section 4 defines well-formedness to allow us to formally state the Optimal Well-formed Incidence Selection Problem (Definition 5) and Section 5 summarizes the algorithm of [19] for solving the problem. Section 6 then combines the elements of the previous sections to derive and validate a new algorithm for generating optimal well-formed systems of rigid body incidences. The diagram, Figure 7, p. 15, summarizes the relations established here compared to the earlier papers, and is a *recommended reading companion*.

1.1. *Background*

Large algebraic systems arise, for example, in industrial geometric constraint solving [3, 4, 7, 10, 18]. Modern solvers employ sophisticated recursive decompositions into rigid subsystems and then recombine the rigid sub-solutions back into a global solution. The recombination systems consist of constraints (that assert incidences between *shared objects*, i.e. copies of objects appearing in different rigid subsystems in the decomposition). In theory, even if the initial system is *well-constrained*, i.e. generically has at least one and at most finitely many realizations, collecting all the incidence constraints between shared objects typically yields redundant, but consistently overconstrained recombination systems. In practice, however, recombination systems need to be solved by generating finite precision intermediate solutions for the subsystems. The incidence constraints due to shared objects then appear in perturbed form in the sharing subsystems and the consistency of the redundant or dependent constraints is difficult to track or verify.

Industrial-size problems are automatically, often combinatorially, decomposed. Selecting an optimal recombination system without ensuring that it is well-formed risks including redundant constraints and excluding essential constraints. The latter will clearly result in wrong output. On the other hand, the entire recombination system, while typically redundant is originally consistently overconstrained. But when perturbed it becomes inconsistent, also resulting the wrong output. Numerical solvers and algebraic solvers that rely on exact equivalence return no solution. Bézier subdivision solvers return solution intervals and are therefore more robust to perturbations (see e.g. [15, 16, 22]). But, in practice, finding the right set of tolerances to capture all solution intervals is tricky. Therefore, to ensure robustness against such numerical errors, reliable solvers of recombination systems need an algorithm to select a *well-formed* subset of incidences, especially when the input is well-constrained.

A second problem for industrial solvers is that the recombination systems are highly nonlinear and have too many variables. This complexity is often not intrinsic and careful analysis shows that such systems can be reformulated or *parametrized* (see e.g. [2, 3, 14, 21]) to yield much a smaller recombination system. Effective solvers of recombination systems therefore need an algorithm that finds a parametrization that minimizes the algebraic complexity.

Recently, progress has been made on both fronts. The *Well-formed Incidences Algorithm* in [19] is purely combinatorial. It generates well-formed recombination systems for collections of incident rigid bodies, in the sense that it selects equations whose roots are a small superset (of perturbations) of the roots of the original system. In particular, for well-constrained collections, it selects a system of independent equations that have finitely many solutions, and for inconsistently overconstrained collections it selects a system of equations with no solution. The original constraints are then used to eliminate extraneous answers. The algorithm avoids numerical or algebraic treatment by exploiting the combinatorial structure of the incidences, specifically, two underlying matroids. Unfortunately, its output system is in general not optimal with respect to algebraic complexity. Conversely, the algorithm in [20] optimizes the algebraic complexity of recombination systems of incidences for collections of rigid bodies. By recognizing situations where known rational parametrizations of polynomial systems can be leveraged, it is possible to eliminate variables [2, 21]. In particular, for incidence constraints, the well-known kinematic substitutions used in robotics [3, 14] can be applied. Typically though the incidences in our applications form a more general graph of rigid body interactions (Figure 1) than a single chain or cycle of molecular bonds or articulated robotic links. The *Optimal Incidence Tree Algorithm* [20] determines a partial elimination ordering, i.e. an *incidence tree*, that minimizes first the number of variables and then the degree in the rationally parametrized recombination system. The algorithm exploits the underlying cycle matroid of the so-called overlap graph of the rigid body collection. For a large class of so-called *standard* collections, the system output by the algorithm is a much smaller one than the original recombination system with provably optimal algebraic complexity (within the class of incidence tree parametrized systems). This makes many practical problems solvable for the first time. Unfortunately, however, this algorithm does not guarantee a well-formed recombination system, free of generic, combinatorial dependencies.

1.2. *Scope*

Most geometric constraint systems are quadratic. Linearizing by taking the Jacobian, followed by random instantiation and rank determination can determine with high probability whether the overall system is generically dependent; but this does not help to determine the structure of generic dependencies, organize all generically maximally independent systems or optimize over these systems (the task of this paper). This issue arises already in the setting of large linear systems, leading to matroid methods and generic concepts such as structural rank. This is one of the historical motivations for combinatorial and matroid approaches to geometric constraint solving.

Secondly, note that without genericity assumptions, to deal with the problems addressed in this paper, one needs a full automated incidence geometry theorem prover to detect all dependencies. No combinatorial method could possibly work, nor would any linear algebra based method such as mentioned above. The underlying problem is computationally as hard as polynomial ideal membership, must use algebraic computation from the start and has double exponential complexity. Genericity assumptions permit combinatorial preprocessing to reduce the algebraic complexity.

Thirdly, we restrict the scope to incidence constraints only since even in two dimensions, no provably correct combinatorial characterization or algorithm for angle and incidence constraints systems is known (A partial solution is given in [23]).

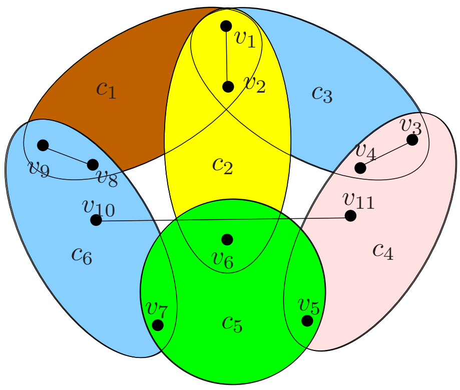

Fig. 1. **Example 1**: Diagram of a collection C of six rigid bodies c_i. Line segments in this diagram represent fixed distances or alternatively rotationally symmetric rigid bodies and shared points represent incidence constraints. The collection is generically rigid in $3D$.

2. Resolving Collections of Rigid Bodies

In this section we formally define a *collection of maximal rigid bodies* D and the corresponding system of incidences $\mathbf{I}(D)$. Figure 1 shows a collection C of six rigid bodies c_i. They are constrained by *incidences or shared points* v_i as shown and there is a fixed or given distance constraining the points v_{10} and v_{11}. Alternatively, the fixed length line segment between v_{10} and v_{11} can be thought of as a seventh, rotationally symmetric rigid body and all the constraints are incidences. Obtaining a *realization* or *resolution* of C means fixing a coordinate system, say that of c_1, and repositioning c_2, c_3, c_4, c_5, c_6 in the coordinate system of c_1, in such a way that the incidences are satisfied. Let $\mathbf{x}_{i,c_j} \in \mathbb{R}^3$ be the coordinates of v_i in c_j's *local* coordinate system. Given these local coordinates $\mathbf{x}_{i,c_j}$, we can, for example, formulate the challenge as determining (real) translations $\mathbf{t}_j \in \mathbb{R}^3$ and the six free (real) parameters of a symmetric 3×3 matrix $\mathbf{M}_j$ representing the composition of three rotations so that for all points v_i in c_1 and rigid bodies c_j $j > 2$, and for for all the points v_i shared by the rigid bodies c_j and c_k, $j, k \neq 1$,

$$\begin{aligned} \mathbf{x}_{i,c_1} &= \mathbf{M}_j \mathbf{x}_{i,c_j} + \mathbf{t}_j, \\ \mathbf{M}_j \mathbf{x}_{i,c_j} + \mathbf{t}_j &= \mathbf{M}_k \mathbf{x}_{i,c_k} + \mathbf{t}_k. \end{aligned} \tag{1}$$

This reflects the fact that a rigid body in 3D has 3 degrees of freedom (dof) of position and 3 of orientation (but a pair of points obeying a distance constraint has only 5 degrees of freedom and only 3 positional dofs are meaningful for a point):

$$\begin{aligned} dof(v) &= 3 \text{ for any point } v, \\ dof(e) &= 5 \text{ for any pair of points } e \text{ with a fixed distance between them} \\ &\quad \text{for example if both points belong to the same rigid body.,} \\ dof(c_i) &= 6 \text{ for any rigid body } c_i \text{ unless } c_i \text{ is a vertex or an edge.} \end{aligned}$$

The above matrix equations (1) each split into three scalar incidence equations, short *incidences*, one for each coordinate $l = 1, 2, 3$. When v_i is a point shared by c_j and c_k, we identify the

$$\text{incidence as } (v_i, \{c_j, c_k\}, l) \tag{2}$$

A collection C of rigid bodies is *rigid* if it generically has at most finitely many realizations (after factoring out trivial motions. Thus well-constrained collections are rigid and generically have at least one realization.) If we represent C as a hypergraph, with the shared points being the vertices and the rigid bodies c_i the hyperedges, then C is **generically rigid** if all non-degenerate geometric realizations with the same hypergraph as C are rigid. Here degenerate is defined as the zero set of some degeneracy polynomial. For standard definitions of genericity in combinatorial rigidity see [6].

Now we are ready to define a *collection of maximal rigid bodies*. Its properties are central to all three combinatorial algorithms in this paper; and are naturally obtained as the output of other existing algorithms for geometric constraint decomposition and recombination [18].

Definition 1 (collection of maximal rigid bodies). Let X be the (coordinate free) points shared by a collection of rigid bodies $C := \{c_1, \dots\}$ in three dimensions. The pair $D := (X, C)$ is a *collection of maximal rigid bodies* if the following hold.
(i) The rigid bodies overlap in at most 2 points: for $i \neq j$, $c_i \bigcap c_j \subseteq X$, $|c_i \bigcap c_j| \leq 2$.
(ii) The rigid bodies are *distinct* with respect to X: for every c_i and c_j, c_i contains at least one point in X that is not shared by c_j.
(iii) The rigid bodies in C form a *covering set* for X: every point in X lies in at least one rigid body in C.
(iv) The only generically rigid proper sub-collections are single rigid bodies: there is no generically rigid subcollection of at least two rigid bodies that covers only a proper subset of X. (This condition says that each rigid body is *maximal* over proper subsets of X.)

A shared point $v \in X$ has local coordinates with respect to each of the rigid bodies containing it. However, in this paper, we need not care about the actual geometry (size, shape, etc.) of the rigid bodies c_i but treat them generically, i.e., as subsets of X.

In combinatorial rigidity, collections of incident rigid bodies are called *body-multi-pin* systems [11]. We note that in general, testing Property (iv) or generic rigidity of body-multi-pin systems is at least as hard as the famous open problem of testing rigidity of 3D bar-and-joint systems. In practice, however, as explained in the Introduction, collections of maximal rigid bodies naturally occur during the recombination of automatically decomposed geometric constraint systems [18]. So we need not test Property (iv) in general but merely assume it holds during the recombination phase of typical decomposition-recombination algorithms. This is a reasonable assumption for algorithms such as [17] and inputs from the mentioned practical CAD applications.

Furthermore, Property (iv) is not necessary for the proofs in this paper. It becomes necessary only if one wants to *use* the results in this paper to guarantee an (A) optimally parametrized system that (B) contains exactly all necessary constraints. [20] explains the role of the requirements, especially (iv), in guaranteeing (A). For optimal parametrizations, a further requirement of completeness is needed, leading to the definition of *standard collections*. We discuss the role of the requirements in guaranteeing (B) later, immediately following the formal definition of a well-formed system.

In CAD applications, each rigid body - and the positions of points in its local coordinate system - are the result of numerically solving polynomial systems and hence can contain error. In particular, the distance between two shared points in one of the sharing bodies may not equal the distance in the other sharing body. In these cases, the complete set of incidence equations given above would be inconsistent. (There are more incidence equations than unknowns and some incidence equations differ only by numerical roundoff.) The entire set of incidence equations is not well-formed even if the collection is generically rigid.

However, these dependencies are of very specific types. For example, two shared points *should* have the same distance in all sharing bodies and hence 5 of their coordinate incidences should effectively imply the 6th. This observation leads to the definition of well-formed systems of incidences in [19] and Section 4. Since arbitrarily long minimal cycles of such dependencies can occur, well-formed systems of incidences are nontrivial to find.

3. Overlap Graphs and Optimal Elimination of Incidences

The Optimal Incidence Tree Algorithm of [20] takes as input a collection of maximal rigid bodies H and outputs subcollection of maximal rigid bodies $D := (X, C)$, along with a subset S_D of incidences partially ordered for elimination. S_D is a spanning tree and is called *incidence tree*. If H is *standard*, i.e. additionally satisfies a completeness property, then elimination of incidences using this tree and rational parametrizations based on quaternions, leaves a parametrized system of optimal algebraic complexity (number of variables and degree)[20]. The optimization of algebraic complexity, over this large class of rational *incidence-tree parametrizations*, is computed with the help of the overlap graph (cf. the entry on the *left* in Figure 7) .

Definition 2 ([20] overlap graph). An *overlap* graph $\mathcal{O}_D$ of a collection of maximal rigid bodies $D := (X, C)$ is a weighted completed undirected graph whose vertex set $V_D = C$ are the rigid bodies c_j and whose edge set E_D represents incidences between pairs of rigid bodies. The edge weights are $w_D(c_i, c_j) := \begin{cases} 6 & \text{if } |c_i \cap c_j| = 0, \\ 3 & \text{if } |c_i \cap c_j| = 1, \\ 1 & \text{if } |c_i \cap c_j| = 2. \end{cases}$

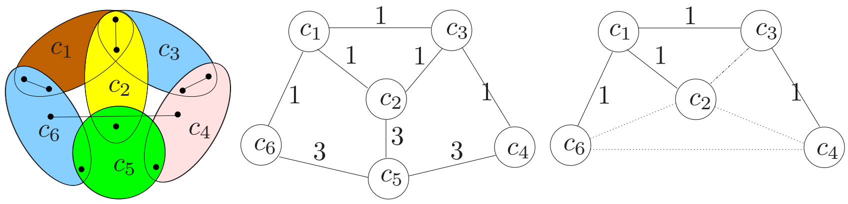

Fig. 2. (*left*) Example 1 of Figure 1, (*middle*) its weighted **overlap graph** and (*right*) the minimal **spanning tree** S_D of one covering set (the points of c_5 are covered by c_2, c_4 and c_6).

The weight of each overlap graph edge represents the number of incidences associated with that edge that would remain after elimination: one incidence for each pair of points shared by two rigid bodies and three incidences for each single point that is shared by two rigid bodies but not as part of a shared pair. Therefore an edge of weight k in the overlap graph can be viewed as the result of applying or eliminating $6-k$ (single coordinate) incidence constraints. The set of incidences obtained from all the overlap edges in the spanning tree S_D is called the set of *elimination incidences*. If a spanning tree edge in S_D denotes a shared pair of points, the corresponding 5 (out of 6 possible) elimination incidences are chosen arbitrarily.

The Optimal Incidence Tree Algorithm chooses an optimal covering set of the rigid bodies and then an optimal, rooted spanning tree S_D of the overlap graph $\mathcal{O}_D$ (see Figure 2, *middle*) of D that, in particular, minimizes the sum of edge weights. Since the input is a collection of maximal rigid bodies, the optimization over covering sets can be done efficiently. Example 1 has 30 independent variables (because it has six rigid bodies, one of which can be fixed arbitrarily). Applying the elimination (parametrization) suggested by Optimal Incidence Tree Algorithm reduces the realization problem from 30 equations in 30 variables to 8 equations in 8 variables.

4. Well-formed Incidence Selection

The Optimal Incidence Tree Algorithm determines an incidence tree whose elimination optimizes complexity, but, as Figure 3, page 11 of Example 1 will illustrate, the remaining parametrized system of incidences typically has more equations than variables even if the original collection was well-constrained. The key challenge is to pick an incidence tree and a remaining parametrized system such that number of variables matches the number of equations; and so that it includes essential constraints but avoids redundant constraints. If the solutions of the resulting system form a finite set that includes all the solutions of the original system, we call it *well-formed*. To make this notion precise, we need the following definitions.

Definition 3 (rdof and local incidence cycle). (i) Let T be any subset of rigid bodies in a collection of maximal rigid bodies $D := (X, C)$. The *removed* degrees of freedom are obtained from the simple inclusion-exclusion formula for the degrees of freedom of the union of rigid bodies in T:

$$\mathrm{rdof}(D,T) := \sum_{Q \subseteq T, |Q|>1} (-1)^{|Q|} \, dof\big(\bigcap_{c_i \in Q} c_i\big)$$

The *rdof* intuitively computes (by inclusion-exclusion) the number of degrees of freedom removed by the shared object incidences in a subcollection Q of a collection of maximal rigid bodies D. As shown in [19], by a simple Moebius inversion together with the fact that the intersections $dof(\bigcap_{c_i \in Q} c_i)$ are of cardinality no more than 2, the above expression reduces to $5 \sum_e (\sharp e - 1) + 3 \sum_v (\sharp v - 1)$, where the first summation is over pairs e of points in X, shared by $\sharp e$ clusters in C and the second summation is over singleton points $v \in X$ shared by $\sharp v$ clusters. If a pair $e = (v_1, v_2)$ of points is shared by 2 clusters then both clusters contribute to $\sharp e$, but neither to $\sharp v_1$ nor $\sharp v_2$.

(ii) Let $\mathbf{I}(D,T)$ be the set of incidences $(v_i, \{c_j, c_\ell\}, p)$ in the set of incidences $\mathbf{I}(D)$ for which $c_j, c_\ell \in T$, v_i is point in X and $p \in \{1,2,3\}$ is a coordinate. A cycle

$$(v_i, \{c_{j_1}, c_{j_2}\}, p), \ldots, (v_i, \{c_{j_{k-1}}, c_{j_k}\}, p), (v_i, \{c_{j_k}, c_{j_1}\}, p)$$

is called a *local incidence cycle* if $k \geq 3$. —□

That is, for a fixed coordinate p, a local incidence cycle connects rigid bodies c_j that share a vertex v_i.

Both expressions in (i) above appear in the combinatorial rigidity literature, for example [8, 9] in addition to [12] and [19]. Although the first has exponentially many terms in $|Q|$, it is conceptually cleaner since it is the direct inclusion-exclusion formula and it permits generalization both for non-standard collections and for higher (than three) dimensions.

Definition 4 (well-formed set of incidences). A set of incidences $\mathbf{I}(D)$ of a collection of maximal rigid bodies D is *well-formed* if it
(a) has no local incidence cycle;
(b) for any $T \subseteq D$, $|\mathbf{I}(D,T)| \leq \mathrm{rdof}(D,T)$;
(c) $|\mathbf{I}(D)| = \mathrm{rdof}(D,D)$.

That is, if (a) and (b) are not met, then even a well-constrained collection could lead to an overconstrained system of incidences and if (c) is not met, not all relevant constraints may be included in the final computation.

This leads to issue (B) concerning the use of Property (iv) that was raised after Definition 1. Property (iv) tells us that $\mathrm{rdof}(D,T) < 6(|T|-1)$ for any subcollection T of at least 2 rigid bodies that covers only a proper subset of X. In fact, Property (iv) can be omitted altogether if the collection is guaranteed to be well-constrained (which the paper stipulates for sake of exposition), because the rdof of a well-constrained collection of maximal rigid bodies can be shown to equal the number of constraints [8, 12]. This continues to be true provided the collection is not overconstrained, i.e., there are no dependent constraints. The rdof count is not useful for detecting if there are dependent constraints; such detection leads to long open problems in combinatorial rigidity including so-called 3D bar-and-joint rigidity [6] and body-multi-pin rigidity [11].

However, the algorithm for obtaining well-formed systems of incidences can also be used for collections with dependent constraints, or collections that are over-constrained as described in [19]. In such cases there are examples (see e.g. [9]) where the rdof does not even provide an upper bound on the number of independent constraints, i.e., a well-formed system could omit necessary constraints if Property (iv) is not satisfied.

We can now formally state the problem of this paper.

Definition 5 (Optimal Well-formed Incidence Selection Problem). Let $D := (X, C)$ be a collection of maximal rigid bodies and S_D the tree of incidences (for elimination) that is output by the Optimal Incidence Tree Algorithm. Give an efficient algorithm for finding a well-formed set of incidences $\mathbf{I}(D)$ that contains the incidences defined by S_D.

The difficulty of picking well-formed systems is illustrated in Figure 3, extending Example 1. Example 1 has one distance constraint (between rigid bodies c_4 and c_6) and we need to select 29 incidence constraints for recombination to have a generically well-constrained system of size 30 by 30. Figure 3, *bottom*, shows two well-formed incidence systems while the selection of Figure 3, *top*,*right*, has dependent incidence constraints and is not well-formed.

5. Seam Graphs and Well-formed Incidence Selection

Seam graphs allow us to pick well-formed incidences. In essence, a seam graph replicates shared points in a collection of maximal rigid bodies, and connects the replicas appropriately by edges (cf. the entry on the *right* in Figure 7) .
Formally, a *seam graph* $\mathcal{G}_D$ of the collection of maximal rigid bodies $D := (X, C)$ is an undirected graph

$$\mathcal{G}_D := (\mathcal{V}_D, \mathcal{E}_D).$$

For each point $v_i \in X$, S_{v_i} is the set of c_j that contain v_i. For each each point $v_i \in X$ and each rigid body c_j in S_{v_i}, the seam graph $\mathcal{G}_D$ contains $\mathcal{V}_{v_i}$, the set of $|S_{v_i}|$ labeled copies $v_{ij} := (v_i, c_j)$ of v_i. We denote

$$\mathcal{V}_D := \bigcup_{v_i \in X} \mathcal{V}_{v_i}.$$

The set $\mathcal{E}_D$ of $\mathcal{G}_D$ consists of two types of edges (see Figure 4): point seam edges and line seam edges. For each original shared point v_i, the *point seam* edges in $\mathcal{PE}_D$ connect every pair of vertices in the set $\mathcal{V}_{v_i}$ (forming a complete graph). For every

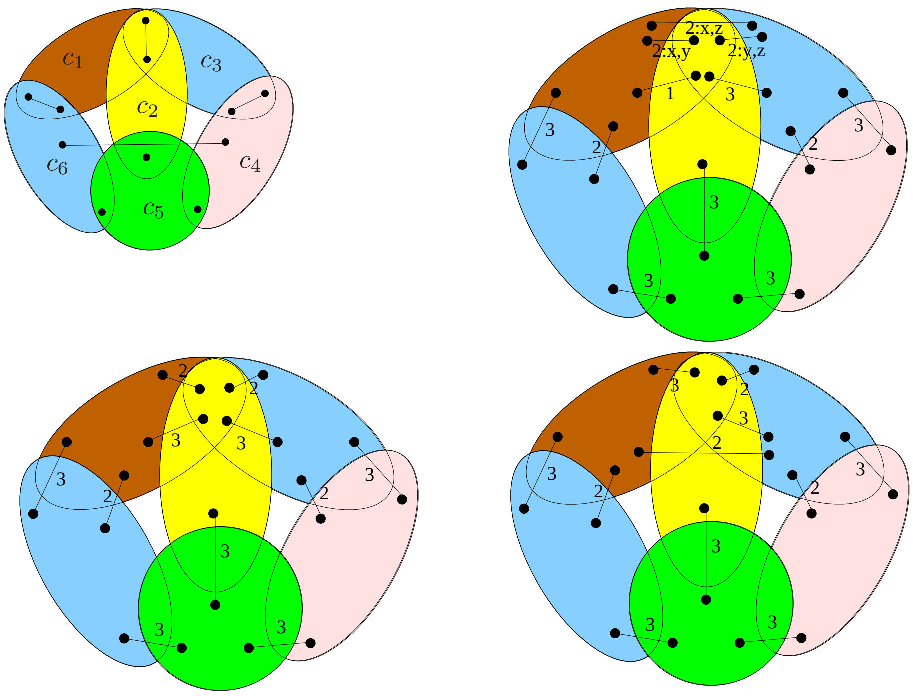

Fig. 3. Choosing **well-formed sets of incidences** for Example 1 (replicated *top*,*left*) is nontrivial. The other three panels show selections of incidence constraints so that the number of variables match the number of constraints. The selections only differ in the top rigid body incidences. A label $2 : x, y$ indicates that of the three possible incidence constraints only the x and y component are selected; a label 1 or 2 means that 1 or 2 randomly selected coordinates are matched; 3 means that all coordinates are matched. While the two selections (*bottom*) are well-formed, the selection (*top*,*right*) is not well-formed, but has a redundant constraint.

pair of points $e := (v_i, v_k)$, $v_i, v_k \in X$, shared by one of $|S_e|$ pairs (c_p, c_j) of rigid bodies, we create a copy for each: $(v_{ip}, v_{kp}) := ((v_i, c_p), (v_k, c_p))$ and $(v_{ij}, v_{kj}) := ((v_i, c_j), (v_k, c_j))$. We denote by E the set of pairs $e := (v_i, v_k)$ and by $\mathcal{E}_e$ the set of edges. In short,

$$\mathcal{E}_D := \mathcal{LE}_D \cup \mathcal{PE}_D; \quad \mathcal{LE}_D := \bigcup_{e \in E_D} \mathcal{E}_e,$$

$$\mathcal{PE}_D := \bigcup_{v_i \in X} \mathcal{PE}_{v_i} := \{(v_{ij}, v_{il}) := ((v_i, c_j), (v_i, c_l)) \in \mathcal{V}_{v_i} \times \mathcal{V}_{v_i}\}.$$

A *seam path* from $u \in \mathcal{V}_v$ to $w \in \mathcal{V}_v$, $u \neq w$, alternates between point and line seam edges: $h_0, g_1, \ldots, h_{2m}, g_{2m+1}, \ldots, h_{4m}$. Each h_{2j} could be empty or consist only of point seam edges. Each g_{2j+1} is a single edge in $\mathcal{LE}_D$ and has a unique partner edge g_{2l+1} such that both edges belong to the same set $\mathcal{E}_e$ for some pair $e \in E_D$. A closed seam path is called *seam cycle*. A *seam forest* consists of all vertices in $\mathcal{V}_D$ and all of the line seam edges in $\mathcal{LE}_D$ but does not contain any seam cycles. A *seam*

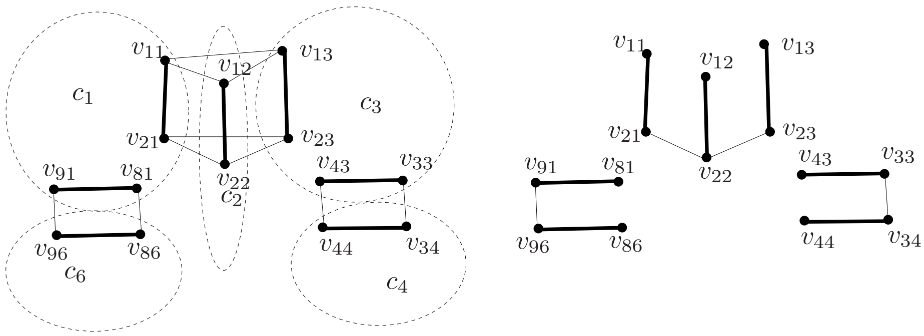

Fig. 4. (*left*) **Seam graph** corresponding to S_D of Example 1 (see Figure 2, page 8) and (*right*) a corresponding **seam tree**.

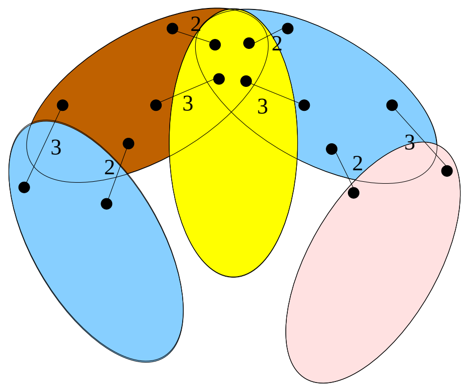

Fig. 5. The **well-formed set of incidences** obtained from the seam tree of Figure 4, *right*.

tree is a seam forest that contains a seam path between every pair of vertices that belong to the same set $\mathcal{V}_v$, for every $v \in X$. Figure 6 shows the seam graph, seam path, seam cycle and seam trees of Example 1. Cycles consisting of only point-seam edges are called *point-seam cycles*. These correspond to the local incidence cycles of Definition 3, where the assignment of coordinate is implicit.

The following theorem shows that seam trees and their point-seam cycle-avoiding maximal completions result in well-formed incidence systems; and that an efficient greedy algorithms exists to find them (cf. the entry on the *top right* in Figure 7) .

Theorem 1 ([19]Well-formed Incidences Algorithm). *Let $\mathcal{G}_D$ be the seam graph of a collection of maximal rigid bodies D. Take a seam forest $\mathcal{F}$ of $\mathcal{G}_D$ and let $\mathcal{F}^*$ be any extension, not necessarily maximal, that avoids point-seam cycles. Construct the set of incidences $\mathbf{I}(D, \mathcal{F}^*)$ as follows. For each point seam edge $((v_i, c_j), (v_i, c_\ell))$ in $\mathcal{F}^*$, add $(v_i, \{c_j, c_\ell\}, 1)$ and $(v_i, \{c_j, c_\ell\}, 2)$ to $\mathbf{I}(D, \mathcal{F}^*)$. If the*

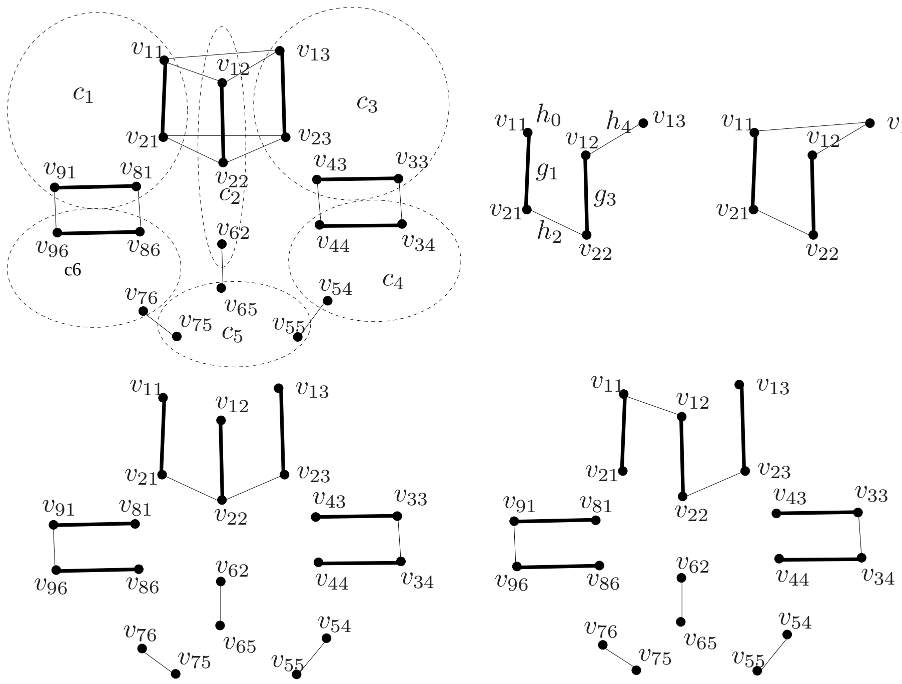

Fig. 6. (*top left*) The **seam graph** of Example 1. (*top,middle*) a **seam path**; (*top,right*) a **seam cycle**; (*bottom*) **seam trees** corresponding to the two bottom choices in Fig.3.

seam edge is in the subset $\mathcal{F} \subset \mathcal{F}^*$, add additionally to the incidences in $\mathbf{I}(D, \mathcal{F}^*)$ the coordinate incidence $(v_i, \{c_j, c_\ell\}, 3)$. Then the following hold.

(1) If $\mathcal{F}^*$ is a seam tree, $\mathbf{I}(D, \mathcal{F}^*)$ is well-formed.
(2) If $\mathcal{F}$ is a seam forest, then $\mathbf{I}(D, \mathcal{F}^*)$ satisfies properties (a) and (b) of Definition 4.
(3) There is a greedy algorithm to complete any seam forest $\mathcal{F}$ into a seam tree $\mathcal{T}$ containing $\mathcal{F}$, in time at most $O(|\mathcal{G}_D|) = O(|V_D + E_D||D|)$; there is a straightforward algorithm to complete any point-seam cycle-avoiding extension of $\mathcal{T}$ into a maximal such extension $\mathcal{F}^*$ in time at most $O(|V_D||D|)$.

To show that well-formed sets of incidences obtained from seam trees may not yield elimination incidences of minimal algebraic complexity, we need to formalize how the seam graph-based selection of incidences corresponds to edges of a spanning tree of an overlap graph (and hence the choice of elimination incidences). This correspondence is given in the next section.

6. Optimal, Well-formed Incidence Selection

To solve the Optimal Well-formed Incidence Selection Problem (Definition 5, page 10), our algorithm starts out by directly using the Optimal Incidence Tree Algorithm of [20]. I.e., it picks a covering set, and the *corresponding* collection of maximal rigid bodies D, such that a spanning tree S_D of the overlap graph $\mathcal{O}_D$ yields a set of incidences whose elimination optimizes (over all covering sets and spanning trees) the algebraic complexity of the remaining system of incidences. In this paper, for ease of exposition, we *directly* deal with this collection of maximal rigid bodies D. The core challenge addressed here is to show that this set of spanning tree incidences can be extended to yield a well-formed system of incidences $\mathbf{I}(D)$, and to exhibit an efficient greedy method to do so. In fact, we will show that this extension is possible for *any* such set of spanning tree incidences, not necessarily the optimal one. This requires us to draw a clear, formal correspondence between the overlap graph $\mathcal{O}_D$ and the seam graph $\mathcal{G}_D$.

We carefully define maps from the cycle matroid on the overlap graph to the seam cycle matroid and point-seam cycle matroid of the seam graph; and the reverse direction. We then proceed to show that the maps preserve independent sets and ensure a containment property that together are sufficient for greedy extensibility into a well-formed set of incidences. I.e., we prove that the maps take the spanning tree in the overlap graph to a seam forest and a point-seam forest containing it, both in the seam graph (lowest, curved solid arrow in Figure 7). A greedy algorithm for obtaining a well-formed set of incidences then follows from the matroid structure of the seam cycle and point-seam cycle matroids and earlier results about well-formed systems of incidences.

6.1. *Correspondence of edges of the spanning tree of the overlap graph to the edges of the seam graph*

In the following, let $D := (X, C)$ be the collection of maximal rigid bodies, $\mathcal{G}_D$ its seam graph and S_D the spanning tree of the overlap graph $\mathcal{O}_D$ output by the Optimal Incidence Tree Algorithm of [20] (cf. the arrow in the center of Figure 7, relating S_D to $\mathcal{G}_D$) .

For the **correspondence of edges in S_D to edges in $\mathcal{G}_D$,** we relate edges of S_D to edges of $\mathcal{G}_D$ by three or five incidence constraints according to the edge weight (type of overlap).

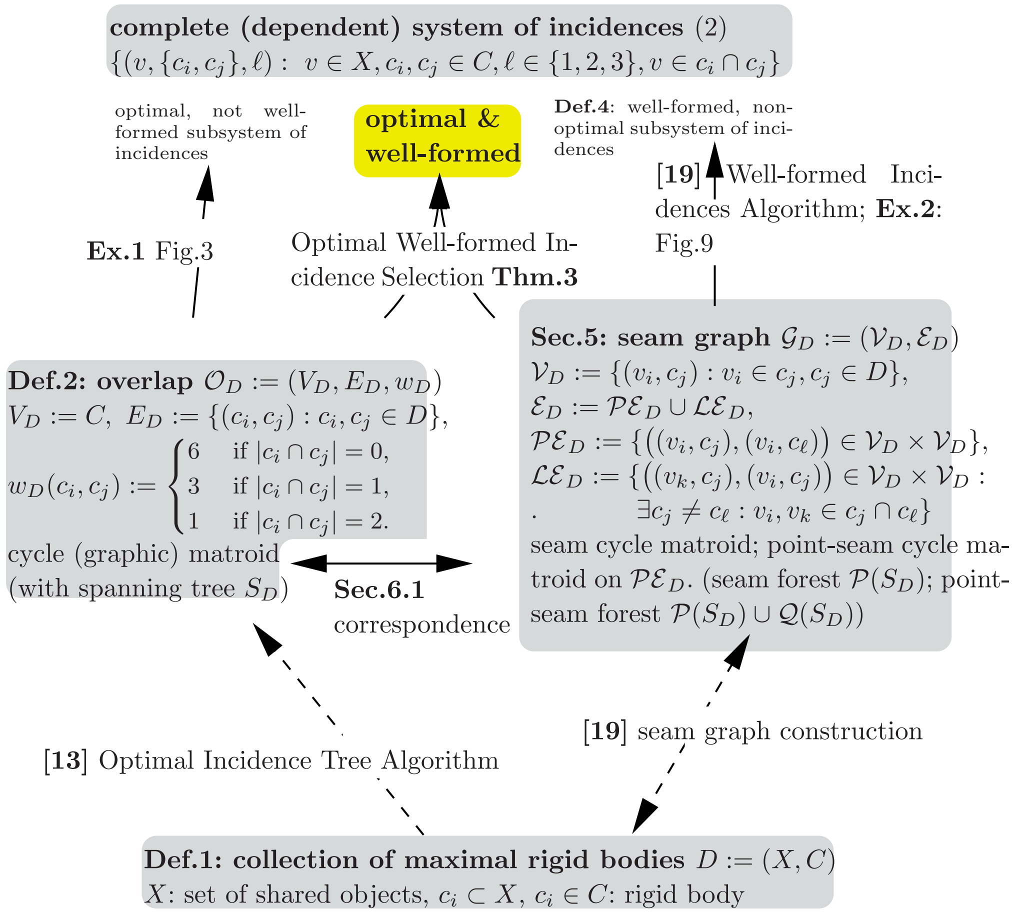

Fig. 7. **The Optimal Well-formed Incidence Selection Algorithm.** To reduce the complete (dependent) system of incidences (*top*) of a collection of maximal rigid bodies $D := (X, C)$ (*bottom*) to an optimal well-formed collection of incidences (*center*), i.e. a smaller system of generically independent equations, the spanning tree S_D (*left*) generated by the (matroid) Optimal Incidence Tree Algorithm of [19] is extended via the seam graph (*right*) of the Well-formed Incidences Algorithm of [13].

Correspondence of edges of S_D to edges of $\mathcal{G}_D$:

weight of edge $\overline{c_j c_\ell}$ in S_D	incidence constraint	$\mathcal{G}_D$ edge
3	$(v_i, \{c_j, c_\ell\}, 1), (v_i, \{c_j, c_\ell\}, 2), (v_i, \{c_j, c_\ell\}, 3)$	$(v_{ij}, v_{i\ell})$
1	$\begin{cases} (v_i, \{c_j, c_\ell\}, 1), (v_i, \{c_j, c_\ell\}, 2), (v_i, \{c_j, c_\ell\}, 3) \\ (v_k, \{c_j, c_\ell\}, 1), (v_k, \{c_j, c_\ell\}, 2) \end{cases}$	$\begin{cases} (v_{ij}, v_{i\ell}) \\ (v_{kj}, v_{k\ell}) \end{cases}$

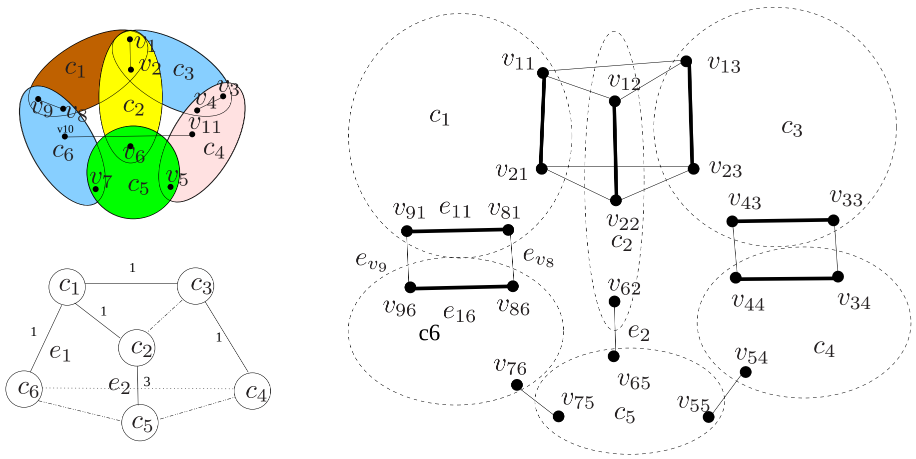

Fig. 8. (*left, top*) Example 1. (*left, bottom*): one **spanning tree** of its weighted overlap graph; (*right*): the **seam graph**.

We call the triple $(v_i, \{c_j, c_\ell\}, 1), (v_i, \{c_j, c_\ell\}, 2), (v_i, \{c_j, c_\ell\}, 3)$ a *triple incidence* and denote by

$$\mathcal{Q}(S_D), \text{ the set of point seam edges in } \mathcal{G}_D \text{ associated with triple incidences.}$$

We call the tuple $(v_k, \{c_j, c_\ell\}, 1), (v_k, \{c_j, c_\ell\}, 2)$ a *double incidence* and denote by

$$\mathcal{P}(S_D), \text{ the set of point seam edges in } \mathcal{G}_D \text{ associated with double incidences.}$$

That is, when c_j and c_ℓ overlap exactly in one vertex v_i and therefore the S_D edge $\overline{c_j c_\ell}$ has weight 3, then the map to the seam edge $(v_{ij}, v_{i\ell}) \in \mathcal{Q}(S_D)$ represents a triple incidence. If c_j and c_ℓ overlap on a fixed pair (v_i, v_k) and therefore the S_D edge $\overline{c_j c_\ell}$ has weight 1, then the map to the seam edges $(v_{ij}, v_{i\ell}) \in \mathcal{Q}(S_D)$ and $(v_{kj}, v_{k\ell}) \in \mathcal{P}(S_D)$ represents one triple incidence and one double incidence (of only the 1st and 2nd coordinate; as mentioned in Section 3, the choice of v_i versus v_k - i.e., the choice of 5 of 6 possible incidences to pick - is arbitrary).

We also have the **reverse correspondence of edges in $\mathcal{G}_D$ to edges in S_D.** Collapse all $\mathcal{G}_D$ vertices that belong to a single rigid body into one. This collapse may map two point seam edges (u_{ki}, u_{kj}) and (v_{ki}, v_{kj}) (at the ends of the line seam edge pair (u_{ki}, v_{ki}) and (u_{kj}, v_{kj})) of $\mathcal{G}_D$ into one edge (c_i, c_j) in S_D of weight 1. Any point seam edge (v_{ki}, v_{kj}), that is not associated with any line seam edge pair, is mapped into the edge (c_i, c_j) of weight 3 of the overlap graph of D.

For an example, consider Figure 8. The edge e_2 in the spanning tree is mapped to the point seam edge e_2 in the seam graph. The edge e_1 in the spanning tree is mapped to the line seam edges e_{11} and e_{16} and point seam edges e_{v_8} and e_{v_9} ($e_{v_8} \in \mathcal{Q}(S_D)$, $e_{v_9} \in \mathcal{P}(S_D)$ or $e_{v_9} \in \mathcal{Q}(S_D)$, $e_{v_8} \in \mathcal{P}(S_D)$). For the reverse correspondence, the vertices $v_{11}, v_{21}, v_{81}, v_{91}$ of rigid body c_1 in the seam graph are collapsed into

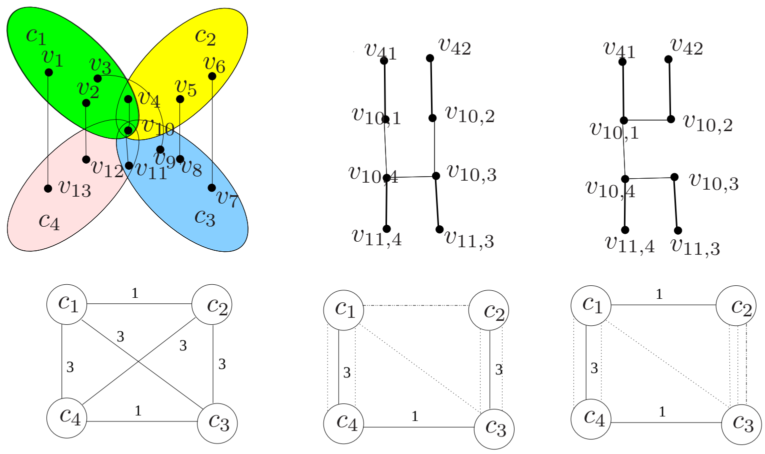

Fig. 9. **Example 2.** (*left bottom*) The overlap graph corresponding to the diagram (*left top*). (*middle*) A seam tree with (*middle*, *top*) a well-formed but non optimal set of incidences and (*middle*, *bottom*) its spanning tree yielding 7 equations. (*right*) A seam tree with a optimal well-formed set of incidences and its spanning tree yielding 5 equations.

one vertex c_1 of the overlap graph. the vertices v_{76}, v_{86}, v_{96} of rigid body c_6 are collapsed into one vertex c_6 of the overlap graph. The reverse correspondence of the two point seam edges e_{v_8} and e_{v_9} results in e_1 in the overlap graph. The vertices v_{12}, v_{22}, v_{62} of rigid body c_2 are collapsed into one vertex c_2 of the overlap graph. The vertices v_{55}, v_{65}, v_{75} of rigid body c_5 are collapsed into one vertex c_5 of the overlap graph. The reverse correspondence of the point seam edge e_2 is e_2 in the overlap graph.

Based on the formalization of this correspondence, we can now give an example that shows that not all well-formed systems of incidences obtained from seam trees optimize algebraic complexity. In Figure 9 (*middle*), the well-formed incidences are not optimal, because the corresponding spanning tree results in 7 equations (the variables are: 1 rotation angle variable between c_3 and c_4, 3 rotation angle variables between c_1 and c_4, 3 rotation angle variables between c_2 and c_3.) On the other hand, the seam tree in Figure 9 (*right*) results in only 5 equations (the variables are: 1 rotation angle variable between c_1 and c_2, 3 rotation angle variables between c_1 and c_4, 1 rotation angle variable between c_4 and c_3).

6.2. *Extension of the spanning tree incidences of the overlap graph to a well-formed set*

We now show that the incidences corresponding to $\mathcal{P} \cup \mathcal{Q}$ can be extended to a well-formed system of incidences $\mathbf{I}(D)$ for collection of maximal rigid bodies D. We do this in two steps. Theorem 2 shows that $\mathcal{P} \cup \mathcal{Q}$ satisfies properties (a) and (b) of Definition 4, i.e. the properties that permit extension to a well-formed system.

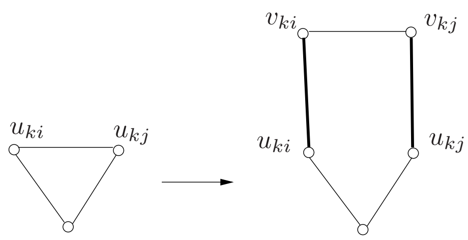

Fig. 10. **Proof of Lemma 1.** Conversion of a point-seam cycle to a seam cycle.

Theorem 3 further gives a simple greedy algorithm to extend $\mathcal{P} \cup \mathcal{Q}$ into a complete well-formed system. Both theorems draw on the following lemma.

Lemma 1. *[seam edges associated with the spanning tree] For a collection of maximal rigid bodies D, the seam edges in $\mathcal{G}_D$ that correspond to edges in the spanning tree S_D of the overlap graph $\mathcal{O}_D$ satisfy:*

(1) The set $\mathcal{Q}(S_D)$ induces a seam forest $\mathcal{F}(S_D)$ in $\mathcal{G}_D$.
(2) The union of $\mathcal{Q}(S_D)$ and $\mathcal{P}(S_D)$ avoids point-seam cycles.
(3) For any completion of $\mathcal{F}(S_D)$ into a seam tree $\mathcal{T}$, $\mathcal{T} \cup \mathcal{P}(S_D)$, is a point-seam cycle-avoiding extension of $\mathcal{T}$.

Proof. For (1), we show that $\mathcal{Q}(S_D)$ avoids seam cycles and hence forms a seam forest. If there is a seam cycle, let $(a, b) := (v_{ki}, v_{kj})$ be a point seam edge in that seam cycle. Then there exists a seam path with endpoints a and b and without the edge (a, b). By reverse correspondence, there is a path between the vertices c_i and c_j in S_D that does not traverse the edge connecting c_i and c_j and the reverse correspondence of (a, b) is an edge that connects c_i and c_j in S_D. This cycle in S_D contradicts its tree property.

For (2), we show that no subset of $\mathcal{P}(S_D) \cup \mathcal{Q}(S_D)$ forms a point-seam cycle. If a subset $\{(v_{k1}, v_{k2}), (v_{k2}, v_{k3}), \ldots (v_{km}, v_{k1})\}$ forms a point-seam cycle in $\mathcal{G}_D$, the reverse correspondences $(c_1, c_2), (c_2, c_3) \ldots$ and (c_m, c_1) form a cycle in S_D, contradicting its tree property.

For (3), by (1) and Theorem 1, we can extend $\mathcal{F}(S_D)$ greedily to a seam tree $\mathcal{T}$. Any extension of $\mathcal{T}$ by $\mathcal{P}(S_D)$ avoids point-seam cycle. If there were a point-seam cycle l, then replacing (u_{ki}, u_{kj}) in the cycle with the unique length 3 seam path connecting its end points in $\mathcal{T}$ (which consists of the two line seam edges (u_{ki}, v_{ki}) and (u_{kj}, v_{kj}) and one point seam edge $(v_{ki}, v_{kj}) \in \mathcal{Q}(S_D)$) would give a seam cycle together with the edges $l \setminus (u_{ki}, u_{kj})$ (Figure 10), contradicting the seam tree property of $\mathcal{T}$. □

Theorem 2. *[well-formedness of $\mathcal{P}(S_D) \cup \mathcal{Q}(S_D)$] Given the collection of maximal rigid bodies D and the minimum spanning tree S_D of the overlap graph of D output* *by the Optimal Incidence Tree Algorithm [20], the proposed system of incidences $\mathcal{P}(S_D) \cup \mathcal{Q}(S_D)$ satisfies properties (a) and (b) of Definition 4 of a well-formed system of incidences for D.*

Proof. Lemma 1 yields a set of edges such that the corresponding incidences (as in Theorem 1) form a partial well-formed system of incidences satisfying properties (a) and (b) of Definition 4. □

We can now state the algorithm that combines well-formedness and optimality for generic collection (X, D) of incidences.

Optimal Well-formed Incidence Selection Algorithm

(1) Determine the optimal spanning tree S_D by the Optimal Incidence Tree Algorithm of Section 3.

(2) Construct $\mathcal{F}^*(S_D)$, the maximal point-seam cycle-avoiding extension of the seam graph corresponding to S_D, as follows.
   - (a) Let $\mathcal{F}^0(S_D)$ be the seam forest formed by $\mathcal{Q}(S_D)$.
   - (b) Compute the seam tree $\mathcal{F}^1(S_D)$ by adding point seam edges to $\mathcal{F}^0(S_D)$ that do not belong to the transitive closure (of the seam paths) of $\mathcal{F}^0(S_D)$ until no more edges can be added.
   - (c) Extend $\mathcal{F}^2(S_D) := \mathcal{F}^1(S_D) \cup \mathcal{P}(S_D)$ greedily to a maximal point-seam cycle-avoiding seam graph $\mathcal{F}^*(S_D)$ by adding the edges from each complete graph $\mathcal{PE}_{v_i}$ of the point seam edges associated with a vertex v_i of V_D.

(3) Convert $\mathcal{F}^*(S_D)$ into the set of incidences $\mathbf{I}(D)$ that defines an optimal well-formed system.
   - (a) For each $v_i \in X$ and each point seam edge $(v_{ij}, v_{i\ell}) \in \mathcal{PE}_{v_i} \bigcap \mathcal{F}^*(S_D)$, add to $\mathbf{I}(D)$ the two incidence constraints $(v_i, \{c_j, c_\ell\}, 1)$ and $(v_i, \{c_j, c_\ell\}, 2)$.
   - (b) For each $v_i \in X$ and each point seam edge $(v_{ij}, v_{i\ell}) \in \mathcal{PE}_{v_i} \bigcap \mathcal{F}^1(S_D)$, add to $\mathbf{I}(D)$ one incidence $(v_i, \{c_j, c_\ell\}, 3)$.

Step (2) proceeds with the next eligible edge without having to backtrack, i.e. the algorithm is greedy. The incidences corresponding to $\mathcal{Q}$ and $\mathcal{P}$ define the optimal partial elimination, while the incidence added in steps (2b) and (2c) define the remaining small, dense core system to be solved directly. The incidences are collected in $\mathbf{I}(D)$.

Theorem 3. *[Properties of the Optimal Well-formed Incidence Selection Algorithm] Given a collection of maximal rigid bodies $D = (X, C)$, the Optimal Well-formed Incidence Selection Algorithm finds an optimized well-formed set of incidences $\mathbf{I}(D)$ in $O(|C|(|C| + |X|)$ time*

By the bounded intersection property (i) of a collection of maximal rigid bodies $|C|$ is at most $O(|X|^3)$, but $|C|$ can be of the same order as or smaller than $|X|$.

Proof. Step 1 outputs a minimum spanning tree S_D that optimizes the algebraic complexity. By Theorem 2, the incidences corresponding to $\mathcal{P}(S_D) \cup \mathcal{Q}(S_D)$ satisfy properties (a) and (b) of well-formed system of incidences. Part 3 of Lemma 1 then guarantees that any greedy extension of $\mathcal{Q}(S_D)$, by completing incidences to form $\mathcal{F}^2(S_D)$, is point-seam cycle-avoiding. Therefore Step 2 extends it to a maximal point-seam cycle-avoiding system. By Part (1) of Theorem 1, this yields a well-formed system of incidences $\mathbf{I}(D)$ for the collection of maximal rigid bodies D that is recovered by Step 3.

The number of vertices in the overlap graph is $O(|C|)$ and the number in the seam graph is $O(|C||X|)$. The optimal recombination algorithm takes at most $O(|C|^2)$ time. Hence the well-formed selection takes $O(|C|(|C| + |X|))$ steps. $\square$

7. Conclusion

Combinatorial preprocessing is an important step to reduce and correctly set up incidence systems so that they can be solved by numeric-algebraic techniques. Graph and matroid based algorithms, in particular, can be very efficient and increase robustness against numerical errors. This step is often applied intuitively, by hand and based on by domain knowledge, when dealing with small problems, but such intuition breaks down for industrial-size incidence problems where subsystems are generated automatically. Merging the outputs of separate combinatorial algorithms to simultaneously satisfy their goals then is a key issue in practice. By analyzing and drawing careful correspondences between the different underlying combinatorial structures, specifically matroids, we were able to derive a new efficient, greedy algorithm combining the advantages of the optimal recombination algorithm of [20] and the well-formed recombination algorithm of [19].

References

1. T. A. Davis. *Direct Methods for Sparse Linear Systems*. SIAM Book Series on the Fundamentals of Algorithms, Philadelphia, 2006.
2. L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics. In *COMPGEOM: Annual ACM Symposium on Computational Geometry*, 2003.
3. I.Z. Emiris, E. Fritzilas, and D. Manocha. Algebraic algorithms for conformational analysis and docking. *Intern. J. Quantum Chemistry, Special Issue on Symbolic algebra in computational chemistry*, 2006.
4. I. Fudos. *Constraint solving for computer aided design.* PhD thesis, Purdue University, Dept of Computer Science, 1995.
5. J. R. Gilbert, C. Moler, and R. Schreiber. Sparse matrices in MATLAB: Design and implementation. *SIAM Journal on Matrix Analysis and Applications*, 13(1):333–356, 1992.
6. Jack E. Graver, Brigitte Servatius, and Herman Servatius. *Combinatorial Rigidity.* Graduate Studies in Math., AMS, 1993.

7. C. M. Hoffmann, A. Lomonosov, and M. Sitharam. Decomposition of geometric constraints systems, part i: performance measures. *Journal of Symbolic Computation*, 31(4), 2001.
8. Bill Jackson and Tibor Jordan. On the rank function of the 3d rigidity matroid. *International Journal of Computational Geometry and Applications*, 16(5-6):415–429, 2006.
9. Bill Jackson and Tibor Jordn. The dress conjectures on rank in the 3-dimensional rigidity matroid. *Advances in Applied Mathematics*, 35(4):355 – 367, 2005.
10. G. Kramer. *Solving Geometric Constraint Systems*. MIT Press, 1992.
11. Audrey Lee-StJohn. *Geometric constraint Systems with applications in CAD and Biology*. PhD thesis, University of Massachussetts, Amherst, MA, USA, 2008. Adviser-Streinu, Ileana.
12. Andrew Lomonosov. Graph and Combinatorial Analysis for Geometric Constraint Graphs. Technical report, Ph.D thesis, Univ. of Florida, Gainesville, Dept. of Computer a nd Information Science, Gainesville, FL, 32611-6120, USA, 2004.
13. J. Peters, J. Fan, M. Sitharam, and Y. Zhou. Elimination in generically rigid 3d geometric constraint systems. In *Proceedings of Algebraic Geometry and Geometric Modeling,Nice, 27-29 Sep 2004*, pages 1–16. Springer Verlag, 2005.
14. M. Raghavan and B. Roth. Inverse kinematics of the general 6R manipulator and related linkages. *J. of Mech. Design, Trans ASME*, 115(3), 1993.
15. Martin Reuter, Tarjei S. Mikkelsen, Evan C. Sherbrooke, Takashi Maekawa, and Nicholas M. Patrikalakis. Solving nonlinear polynomial systems in the barycentric bernstein basis. *The Visual Computer*, 24(3):187–200, 2008.
16. T. Sederberg and T. Nishita. Curve intersection using Bézier clipping. *Computer Aided Design*, 22(9):538–549, 1990.
17. M. Sitharam. Frontier, opensource gnu geometric constraint solver: Version 1 (2001) for general 2d systems; version 2 (2002) for 2d and some 3d systems; version 3 (2003) for general 2d and 3d systems. In *http://www.cise.ufl.edu/∼sitharam, http://www.gnu.org*, 2004.
18. M Sitharam. Combinatorial approaches to geometric constraint solving: problems, progress, directions. In D. Dutta, R. Janardhan, and M. Smid, editors, *AMS-DIMACS book on computer aided and manufacturing*, volume 67, pages 117–163, 2005.
19. M. Sitharam. Characterizing well-formed systems of incidences for resolving collections of rigid bodies. *IJCGA*, 16(5-6):591–615, 2006.
20. Meera Sitharam, Jorg Peters, and Yong Zhou. Optimized parametrization of systems of incidences between rigid bodies. *Journal of Symbolic Computation*, 45:481–498, feb 2010.
21. W. Wang, B. Joe, and R. N. Goldman. Computing quadric surface intersections based on an analysis of plane cubic curves. *Graphical Models*, 64(6):335–367, 2002.

22. C. Yap. Complete subdivision algorithms, I: Intersection of Bézier curves. In *COMPGEOM: Annual ACM Symposium on Computational Geometry*, 2006.
23. Yong Zhou. *Combinatorial decomposition, generic independence and algebraic complexity of geometric constraints systems: applications in biology and engineering*. PhD thesis, University of Florida, Gainesville, FL, USA, 2006. Adviser-Sitharam, Meera.