Some Solved and Unsolved Problems in Subdivision, Surface Parametrization and Algebraic Constraint Elimination

Jörg Peters

SurfLab, U Florida

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collaborators: K. Karčiauskas, U. Reif, M. Sitharam, A. Myles, J. Fan, T. Ni, X. Wu

Geometric Constraint Problem

Parameterization for polynomial equ's
$s^2 + c^2 = 1$ and $\sum_{i=0}^{3} q_i^2 = 1$

$$R := \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \quad Q := \begin{bmatrix} 1-2q_2^2-2q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & 1-2q_1^2-2q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & 1-2q_1^2-2q_2^2 \end{bmatrix}.$$

Challenge: to model and resolve pairwise incidences between rigid bodies for whole networks or graphs of rigid body interactions [SPFZ 2004,2008]
The goal is to find all solutions.

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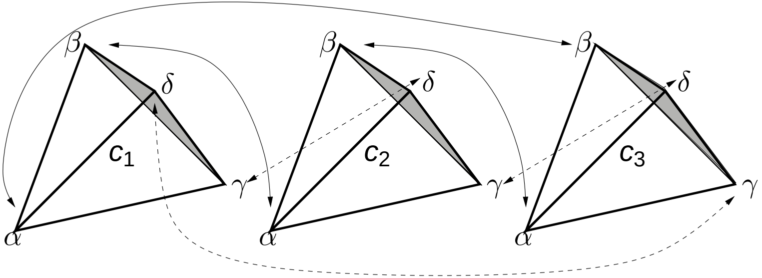
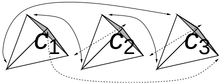
Example: Problem `tetra` – what are all possible configurations?



Three rigid bodies c_i (of 7). Local coordinates $\mathbf{x}_{i,j}$.
Incidences solid, distance dashed.

Unoptimized System



1. all points are covered by $c := \{c_1, c_2, c_3\}$.
2. Pick home coordinate sytem $h := c_1$.
3. Resolve c_2 and c_3 in the coordinates of h ($\mathbf{x}_{i,j}$'s are given; we are solve for M's and T's)

$$\begin{aligned} \mathbf{x}_{1,\beta} &= M_2\mathbf{x}_{2,\alpha} + T_2, \\ M_2\mathbf{x}_{2,\beta} + T_2 &= M_3\mathbf{x}_{3,\alpha} + T_3, \\ M_3\mathbf{x}_{3,\beta} + T_3 &= \mathbf{x}_{1,\alpha} \end{aligned}$$

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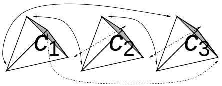

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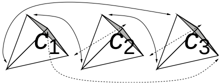

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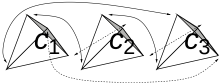

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3 Resolve c_2 and c_3 in the coordinates of h

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$$\vdots$$
$$\|M_3\mathbf{x}_{3,\gamma} + T_3 - \mathbf{x}_{1,\delta}\|^2 = d_3^2,$$

$$M_i := \begin{bmatrix} s_{i2}s_{i3} & s_{i2}c_{i3} & -c_{i2} \\ s_{i1}c_{i2}s_{i3}-c_{i1}c_{i3} & c_{i1}s_{i3}+s_{i1}c_{i2}c_{i3} & s_{i1}s_{i2} \\ s_{i1}c_{i3}+c_{i1}c_{i2}s_{i3} & -s_{i1}s_{i3}+c_{i1}c_{i2}c_{i3} & c_{i1}s_{i2} \end{bmatrix}, \quad T_{c_2} := \begin{bmatrix} p_i \\ q_i \\ r_i \end{bmatrix}.$$

$$s_{i,j}^2 + c_{i,j}^2 = 1.$$

$n = 18$ polynomial equations in n variables
Maximum degree $d = 6$

Simplify

$$\mathbf{x}_{1,\beta} = M_2 \mathbf{x}_{2,\alpha} + T_2$$

Set $\mathbf{x}_{2,\alpha}$ to be the origin in the local coordinates of c_2 and $\mathbf{x}_{3,\beta}$ to be the local origin of c_3: 12 equations.

Parameterize:

R and Q , $\quad s^2 + c^2 = 1$ and $\sum_{i=0}^{3} q_i^2 = 1$

Not 12 variables!

Parameterize by stereographic projection:

$$c := \frac{1-t_0^2}{1+t_0^2},\ s := \frac{2t_0}{1+t_0^2}, \qquad q_0 := \frac{1-\sum_{i=1}^{3} t_i^2}{1+\sum_{i=1}^{3} t_i^2},\ q_j := \frac{2t_j}{1+\sum_{i=1}^{3} t_i^2}, j=1,2,3.$$

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k	incidences	$w(k)$	degree
3	$(a,b,c) \to (a',b',c')$	0	0
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$n = 12$ polynomial equations in the n variables

$$t_{ij}, p_j, q_j, r_j, \text{ for } 1 \le i \le 3, 2 \le j \le 3.$$

maximum degree is $d = 8$

Optimized Parametrization

18 eq's $\longrightarrow$ Parameterization $\longrightarrow$ 12 eq's

12 eq's $\longrightarrow$ Optimal Elimination $\longrightarrow$ 3 eq's

(a) Look at the complete collection of rigid bodies.

(b) partial ordering (tree) of elimination by $w(k)$

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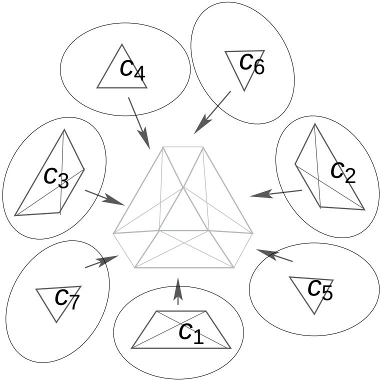
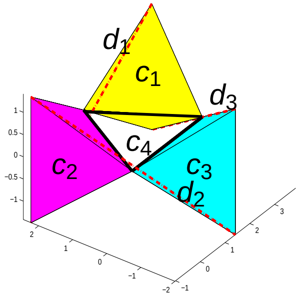
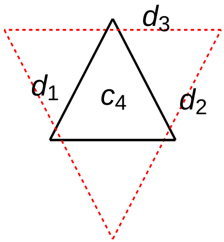


All 7 proper-maximal rigid bodies of Problem `tetra`.

1. Choose $c := \{c_1, c_2, c_3, c_4\}$ as the (not minimal!) covering set.
2. Pick the home coordinate sytem $h := c_4$.
3. Resolve incidences (triangle!) – only 3 distance equations remain.





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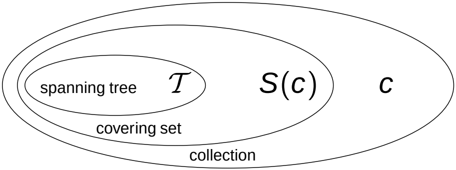
$n = 3$ polynomial equations in the variables t_i.
The maximum degree is $d = 4$.
Solve by [Gaukel 2003].

How to automate Optimal Elimination?



Find the optimal incidence tree
Input: A standard collection of rigid bodies (X, c).
Output: An optimal incidence tree $\mathcal{T}(X, c)$.

(1) Over all covering sets $S(c)$, determine the set $\{\mathcal{T}_\ell\}$ of spanning trees (of the subgraphs of the overlap graph $\mathcal{G}(X, c)$ induced by $S(c)$) of minimum weight $w(k)$

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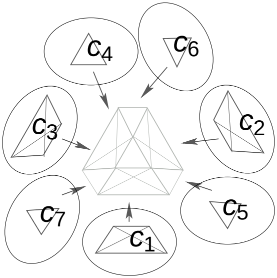

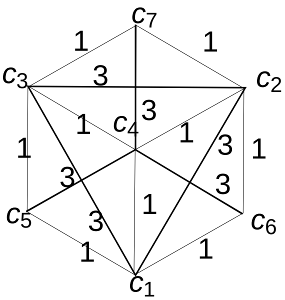

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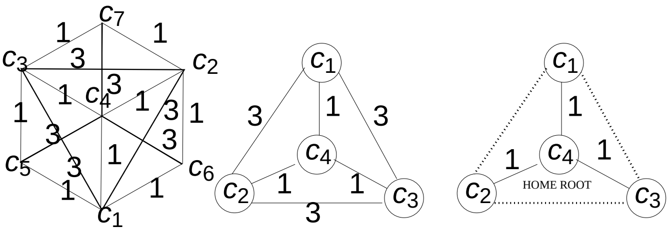

overlap graph   covering set   spanning tree

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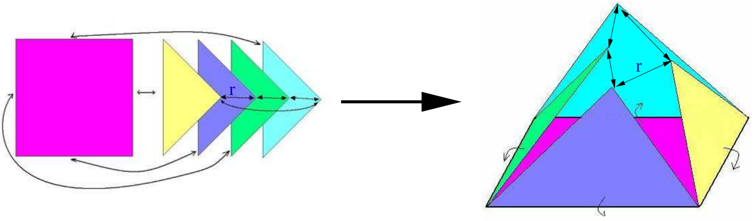
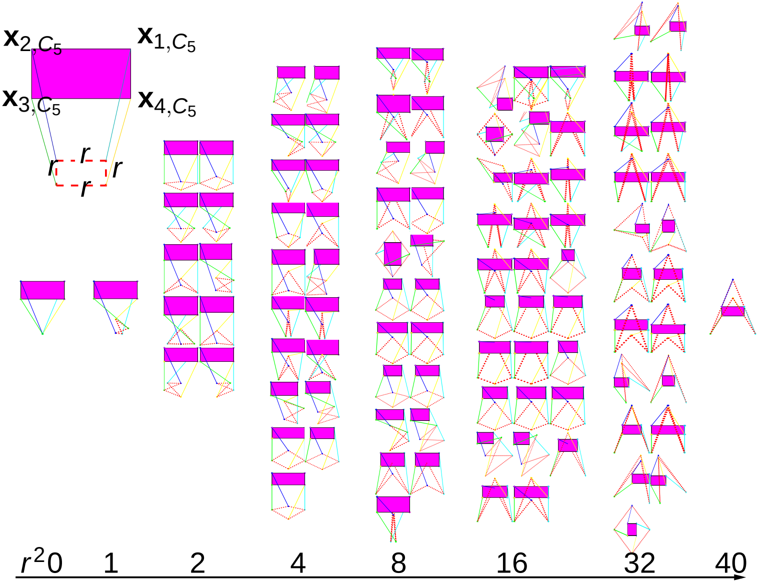
(2) Over all choices of trees in $\{\mathcal{T}_\ell\}$ and roots determine a rooted tree that minimizes the sum of the depths of all nodes.

Geometry Constraints

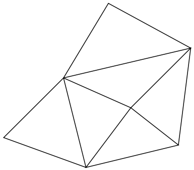


Solution set



bi-3 C^1 interpolation

(in)Famous C^1 cubic interpolation Problem [Schumaker, Alfeld, Lai, $\cdots$]



Given: triangulation and values at the vertices.

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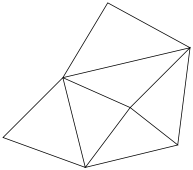
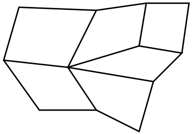
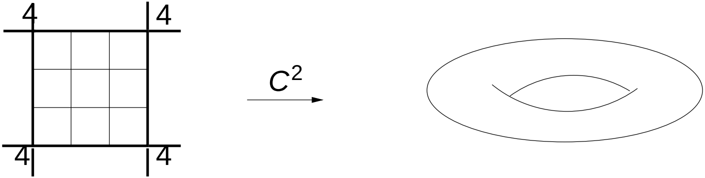




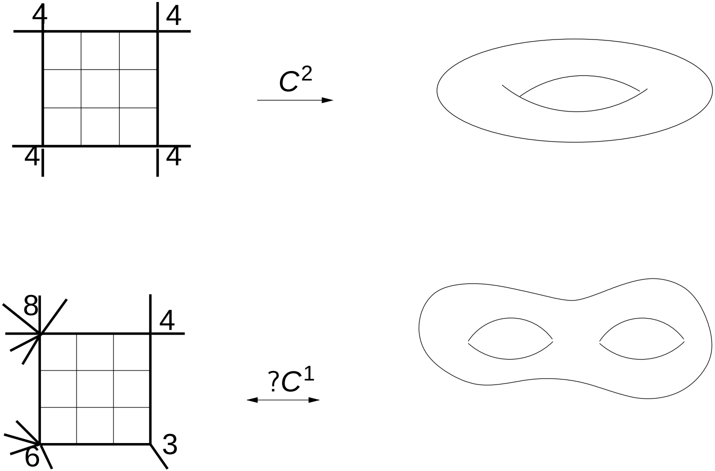
. G^1 (C^1) bicubic interpolation Problem

bi-3 C^1 interpolation



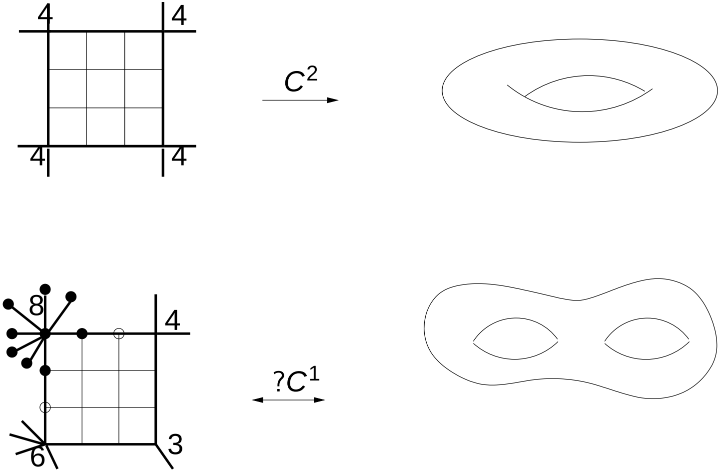
bi-cubic in BB-form; vertex valences = 4, cyclic

bi-3 C^1 interpolation



vertex enclosure problem

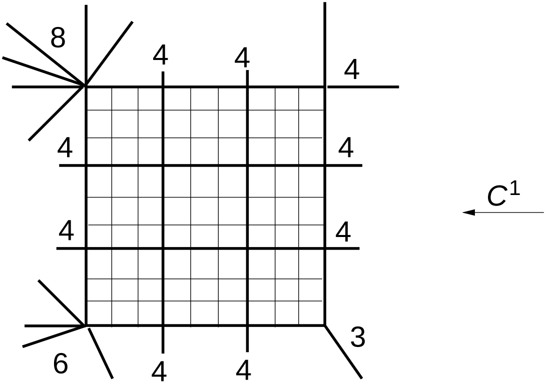
bi-3 C^1 interpolation



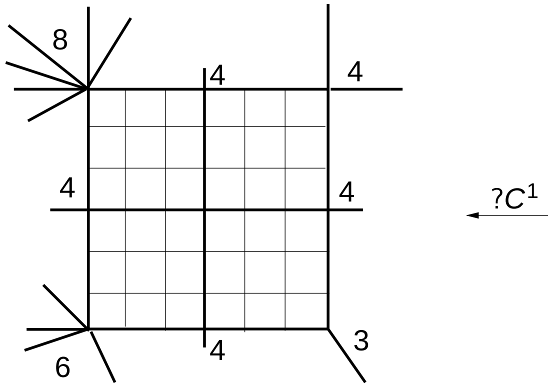
vertex enclosure problem

bi-3 with C^1 continuity

[Fan $\sim$ 2008]



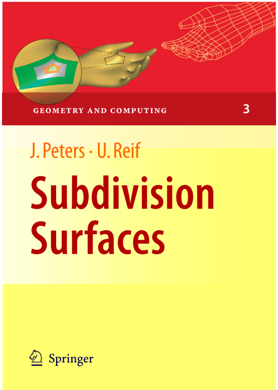
Open Problem: 2×2 bi-3 patches C^1 surface



local or global+structured

~~Subdivision has Problems~~

- High-valence → oscillations. ✓
- Forced Nonconvexity ✓
- Not C^2; diverging curvatures ✓
  Ch 8.1



Problems to work on

- More guide surfaces.
- C^1 bi-3 interpolation (2×2).
- Underconstrained geometric constraint problems.
- What other parameterizations yield useful partial eliminations for a concrete class of problems?