Some Solved and Unsolved Problems in Subdivision, Surface Parametrization and Algebraic Constraint Elimination

Jörg Peters

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Problems in Surface Design and Algebraic Constraints

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Problems of Geometric Constraint Solving

The Unoptimized
Polynomial System of
Problem tetra
Efficient Parametrizations
Optimized Choice and
Partial Ordering of

Problem: C¹ bi-3 nterpolation

Problems in Subdivision Polar Structure Shape Charts Guided Subdivision

Constraint Solving

Parameterization for polynomial equ's $s^2 + c^2 = 1$ and $\sum_{i=0}^{3} q_i^2 = 1$

 $R := \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{smallmatrix} \right], \quad Q := \left[\begin{smallmatrix} 1 - 2q_2^2 - 2q_3^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & 1 - 2q_1^2 - 2q_3^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & 1 - 2q_1^2 - 2q_2^2 \end{smallmatrix} \right].$

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Challenge: to model and resolve pairwise incidences between rigid bodies for whole networks or graphs of rigid body interactions [SPFZ 2004,2008]

The goal is to find all solutions.

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Challenge: to model and resolve pairwise incidences between rigid bodies for whole networks or graphs of rigid body interactions [SPFZ 2004,2008] The goal is to find all solutions.

Example: Problem tetra – what are all possible configurations?

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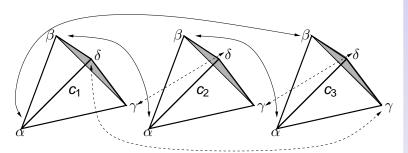
Problems of Geometric Constraint Sol

The Unoptimized Polynomial System of Problem tetra

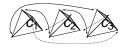
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Problems in Subdivision Polar Structure Shape Charts



Three rigid bodies c_i (of 7). Local coordinates $\mathbf{x}_{i,j}$. Incidences solid, distance dashed.



- 1 all points are covered by $c := \{c_1, c_2, c_3\}$.
- 2 Pick home coordinate system $h := c_1$.
- **3** Resolve c_2 and c_3 in the coordinates of h ($\mathbf{x}_{i,j}$'s are given; we are solve for M's and T's)

$$\mathbf{x}_{1,\beta} = M_2 \mathbf{x}_{2,\alpha} + T_2,$$
 $M_2 \mathbf{x}_{2,\beta} + T_2 = M_3 \mathbf{x}_{3,\alpha} + T_3,$
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$$\|\mathbf{x}_{1,\gamma} - M_2 \mathbf{x}_{2,\delta} + T_2\|^2 = d_1^2,$$

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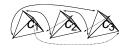
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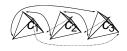
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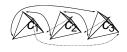
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Guided Subdivision



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The Unoptimized

3 Resolve c₂ and c₃ in the coordinates of h

$$\mathbf{x}_{1,\beta} = M_2 \mathbf{x}_{2,\alpha} + T_2,$$

$$\vdots$$

$$\|M_3 \mathbf{x}_{3,\gamma} + T_3 - \mathbf{x}_{1,\delta}\|^2 = d_3^2,$$

$$M_i := \left[\begin{smallmatrix} s_{i2}s_{i3} & s_{i2}c_{i3} & -c_{i2} \\ s_{i1}c_{i2}s_{i3} - c_{i1}c_{i3} & c_{i1}s_{i3} + s_{i1}c_{i2}c_{i3} & s_{i1}s_{i2} \\ s_{i1}c_{i3} + c_{i1}c_{i2}s_{i3} & -s_{i1}s_{i3} + c_{i1}c_{i2}c_{i3} & c_{i1}s_{i2} \end{smallmatrix}\right], \quad T_{c_2} := \left[\begin{smallmatrix} p_i \\ q_i \\ r_i \end{smallmatrix}\right].$$

$$s_{i,j}^2 + c_{i,j}^2 = 1.$$

n = 18 polynomial equations in n variables Maximum degree d=6

Efficient Parametrizations

$\mathbf{x}_{1,\beta} = M_2 \mathbf{x}_{2,\alpha} + T_2$

Set $\mathbf{x}_{2,\alpha}$ to be the origin in the local coordinates of c_2 and $\mathbf{x}_{3,\beta}$ to be the local origin of c_3 : 12 equations.

Parameterize:

$${\it R}$$
 and ${\it Q}$,

$$s^2 + 0$$

R and Q,
$$s^2 + c^2 = 1$$
 and $\sum_{i=0}^{3} q_i^2 = 1$

Not 12 variables!

Parameterize by stereographic projection:

$$\mathbf{c}\!:=\!\frac{1-t_0^2}{1+t_0^2},\;\mathbf{s}\!:=\!\frac{2t_0}{1+t_0^2},\qquad q_0\!:=\!\frac{1-\sum_{i=1}^3t_i^2}{1+\sum_{i=1}^3t_i^2},\;q_j\!:=\!\frac{2t_j}{1+\sum_{i=1}^3t_i^2}.j\!=\!1,\!2,\!3.$$

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	incidences		degree
3	(a,b,c) o (a',b',c')	0	0
2	(a,b) ightarrow (a',b')	1	2
1	(a) ightarrow (a')	3	4
0	$egin{aligned} (a,b,c) & ightarrow (a',b',c') \ (a,b) & ightarrow (a',b') \ (a) & ightarrow (a') \ \end{aligned}$ none	6	4

Problems of Geometric Constraint Solving

Problem tetra

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n = 12 polynomial equations in the *n* variables

$$t_{ij}, p_j, q_j, r_j, \text{ for } 1 \le i \le 3, 2 \le j \le 3.$$

maximum degree is d = 8

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Problems in Subdivision

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Guided Subdivision

18 eq's → Parameterization → 12 eq's

- 12 eq's → Optimal Elimination → 3 eq's
- (a) Look at the complete collection of rigid bodies
- (a) Look at the <u>complete</u> collection of rigid bodies (b) partial ordering (tree) of elimination by w(k)

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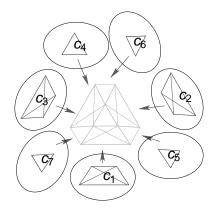
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Problem: C¹ bi-3 Interpolation

Problems in Subdivision Polar Structure Shape Charts

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All 7 proper-maximal rigid bodies of Problem tetra.

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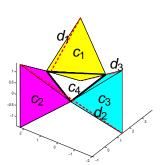
Problems of Geometric Constraint Solving

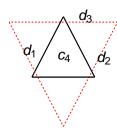
Polynomial System of Problem tetra Efficient Parametrizations Optimized Choice and Partial Ordering of Parametrized Equations

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- **2** Pick the home coordinate system $h := c_4$.
- 3 Resolve incidences (triangle!) only 3 distance equations remain.





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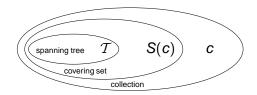
Partial Ordering of Parametrized Equations

1 Choose $c := \{c_1, c_2, c_3, c_4\}$ as the (not minimal!) covering set.

- **2** Pick the home coordinate system $h := c_4$.
- 3 Resolve incidences (triangle!) only 3 distance equations remain.

n=3 polynomial equations in the variables t_i . The maximum degree is d = 4. Solve by [Gaukel 2003].

How to automate Optimal Elimination?



Find the optimal incidence tree

Input: A standard collection of rigid bodies (X, c).

Output: An optimal incidence tree T(X, c).

(1) Over all covering sets S(c), determine the set $\{\mathcal{T}_\ell\}$ of spanning trees (of the subgraphs of the overlap graph $\mathcal{G}(X,c)$ induced by S(c)) of minimum weight w(k)

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Optimized Choice and Partial Ordering of Parametrized Equations

Problems in Subdivision

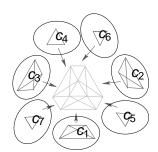
Subdivision
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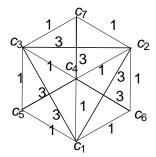
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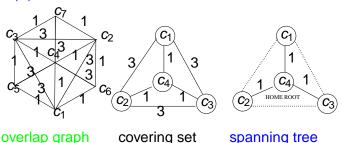
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- (2) Over all choices of trees in $\{\mathcal{T}_{\ell}\}$ and roots determine a rooted tree that minimizes the sum of the depths of all nodes.

Geometry Constraints

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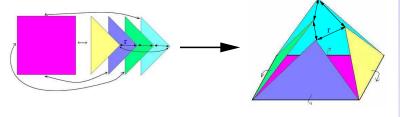
Problem: C¹ bi-3

Interpolation

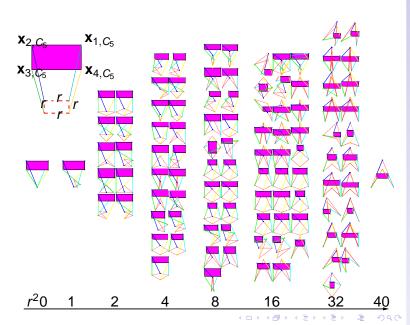
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Solution set



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Polynomial System of Problem tetra Efficient Parametrization

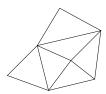
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Problem: C¹ bi-3 Interpolation

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Problem: C1 bi-3 Interpolation

(in)Famous C¹ cubic interpolation Problem [Schumaker, Alfeld, Lai, ...]

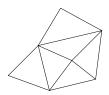


Given: triangulation and values at the vertices.

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Problem: C1 bi-3 Interpolation

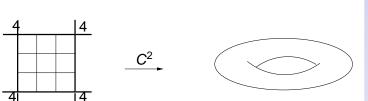
(in)Famous C¹ cubic interpolation Problem [Schumaker, Alfeld, Lai, ...]





G¹ (C¹) bicubic interpolation Problem

bi-3 C¹ interpolation



bi-cubic in BB-form; vertex valences = 4, cyclic

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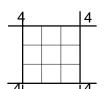
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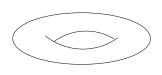
Problem: C¹ bi-3 Interpolation

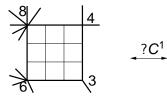
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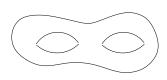
bi-3 C¹ interpolation











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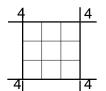
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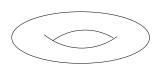
vertex enclosure problem

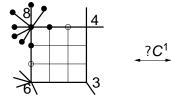


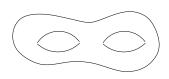
bi-3 C¹ interpolation











Problems in Surface Design and Algebraic Constraints

J Peters

Problems of Geometric Constraint Solving

Polynomial System of Problem tetra Efficient Parametrizations Optimized Choice and Partial Ordering of Parametrized Equations

Problem: C¹ bi-3 Interpolation

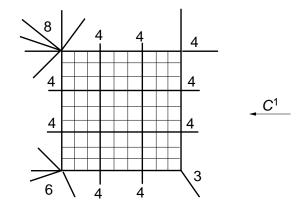
Problems in Subdivision Polar Structure Shape Charts Guided Subdivision

vertex enclosure problem



bi-3 with C^1 continuity

[Fan \sim 2008]



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Problems of Geometric Constraint Solving

Problem tetra
Efficient Parametrization:
Optimized Choice and
Partial Ordering of

Problem: C¹ bi-3 Interpolation

Subdivision
Polar Structure
Shape Charts

Open Problem: 2×2 bi-3 patches C^1 surface

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Polynomial System of Problem tetra Efficient Parametrizations Optimized Choice and

Parametrized Equations

Problem: C¹ bi-3

Problem: C1 bi-3
Interpolation

Subdivision
Polar Structure
Shape Charts

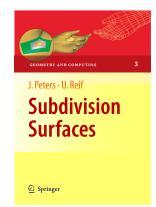
Guided Subdivision

4 4 ?C¹

local or global+structured

Subdivision has Problems

- ► High-valence → oscillations. ✓
- ▶ Forced Nonconvexity √
- Not C²; diverging curvatures √ Ch 8.1



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Problem tetra
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Problem: C¹ bi-3 Interpolation

Subdivision Polar Structure

Guided Subdivision

Guided Subdivision

More guide surfaces.

- ▶ C^1 bi-3 interpolation (2 × 2).
- Underconstrained geometric constraint problems.
- What other parameterizations yield useful partial eliminations for a concrete class of problems?