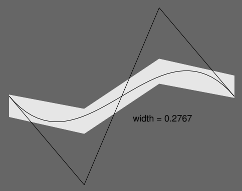
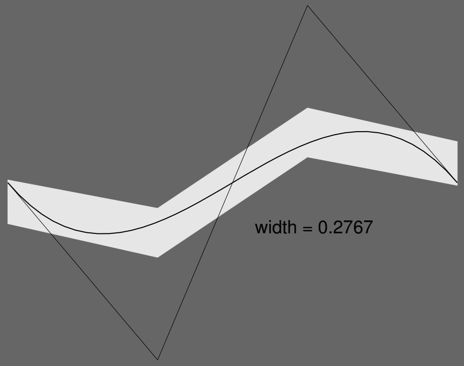
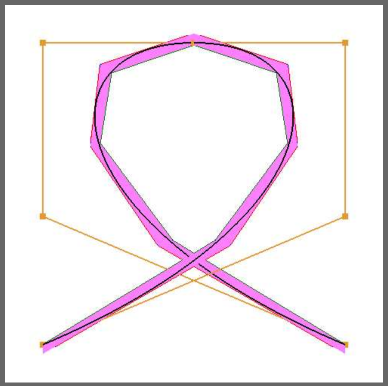
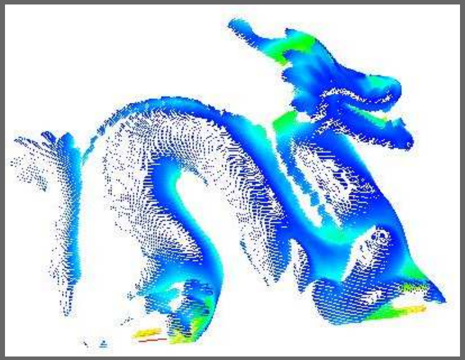
Mid-Structures Linking Curved and Linear Geometry



SIAM Geometric Design, Nov 10–13 2003







Jörg Peters, University of Florida

http://www.cise.ufl.edu/∼jorg

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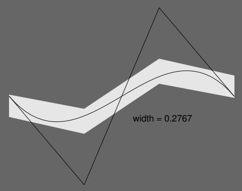
Goals and Outline



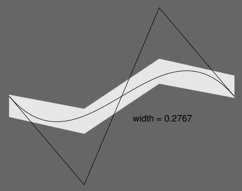
(1) *SLEFEs*: Enclosing Functions

(2) *Mid-structures*: *Quantitatively* Coupling Curved and pw Linear Geometry

(3) *Constrained Design*: One-sided fitting

Improves Robustness (Collision, Rendering, Conversion)

SLEFEs: Enclosing Functions



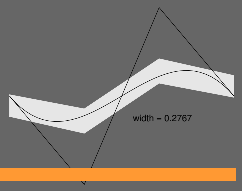
Find an explicit two-sided approximation $\overline{x}, \underline{x}$ of a map x
so that $\underline{x} \leq x \leq \overline{x}$ over the domain of interest.

- *Efficiently* (few pieces, easy to compute)
- *Tightly*
- (predictively) *Refinable*
- (Affinely) *Invariant*, ...

Then the pair $\underline{x}, \overline{x}$ is a

Subdividable Linear Efficient Function Enclosure (SLEFE) of x.

SLEFE construction



Given: space B of functions to be enclosed.

0) *Choose* U (domain of interest), H enclosure functions.

1) $\mathbf{s} := \dim B - \dim B \cap H < \infty$.
Choose s functionals $\mathcal{F} : B \mapsto \mathbb{R}^s$ so that $\ker \mathcal{F} = B \cap H$.

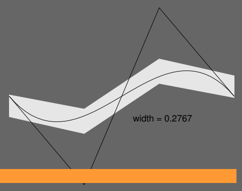
2) Compute new basis $\mathbf{a} : \mathbb{R}^s \mapsto B$ so that $\mathcal{F}\mathbf{a}$ is identity on $\mathbb{R}^s$ and each $\mathbf{a}_\kappa$ matches the same $\dim(B \cap H)$ additional independent constraints.

3) Compute $\underline{\mathbf{a} - L\mathbf{a}} \in H^s$ and $\overline{\mathbf{a} - L\mathbf{a}} \in H^s$ (lower and upper bound)

4) Compute $\mathcal{F}x \in \mathbb{R}^s$ and assemble $\underline{x}$ and $\overline{x}$.

(0),(1),(2),(3) are offline (4) is cheap

Bézier SLEFE Construction

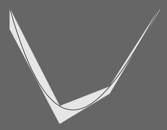


B = degree $d = 3$ polynomial in *Bézier form* $x(u) := \sum_{k=0}^{d} \mathbf{b}_k(u) x_k$

0) U = *[0..1]* (domain of interest),
$H = \mathbf{h}_\mu^m$ hat functions with break points at $\frac{\mu}{m}$, $\mu = 0, \ldots, m$

1) $\dim B - \dim B \cap H = 4 - 2 =$ *2=s*.
Functionals $\mathcal{F} : B \mapsto \mathbb{R}^s$ so that $\ker \mathcal{F} = B \cap H$:
$\mathcal{F}_\nu x := x_{\nu-1} - 2x_\nu + x_{\nu+1}$.

2) New basis $\mathbf{a} : \mathbb{R}^s \mapsto B$:
$\dim(B \cap H) = 2$ constraints: first and last coefficient = 0
coefficients of $\mathbf{a}_1 : (0, -2, -1, 0)/3$, $\mathbf{a}_2 : (0, -1, -2, 0)/3$, $\mathcal{F}\mathbf{a}$ is identity.



3) Compute broken line $\underline{\mathbf{a}_k - L\mathbf{a}_k}$, below and $\overline{\mathbf{a}_k - L\mathbf{a}_k}$ above.

4) Compute $\mathcal{F}x \in \mathbb{R}^s$ and assemble $\underline{x}$ and $\overline{x}$.

(0),(1),(2),(3) are offline (4) is cheap.

[0–3] SLEFE construction: derivation

Given

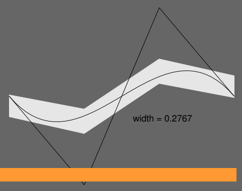
$$p(t) := -\mathbf{b}_1^3(t) + \mathbf{b}_2^3(t), \qquad \mathbf{b}_j^d := \binom{d}{j}(1-t)^{d-j}t^j$$

coefficient sequence $(c_j)_{j=0,..,3} = (0, -1, 1, 0)$.

With $\ell(t) := p(0)(1-t) + p(1)t$ and
$\mathbf{a}_1^3(t) := -\frac{2}{3}\mathbf{b}_1^3(t) - \frac{1}{3}\mathbf{b}_2^3(t), \mathbf{a}_2^3(t) := -\frac{1}{3}\mathbf{b}_1^3(t) - \frac{2}{3}\mathbf{b}_2^3(t),$

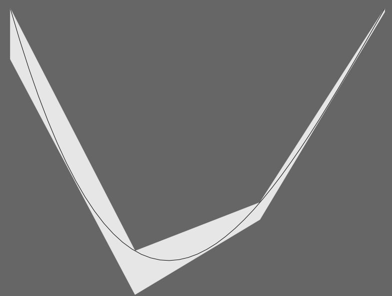
$$p(t) = \ell(t) + 3\mathbf{a}_1^3(t) - 3\mathbf{a}_2^3(t).$$

[0–3] SLEFE construction: derivation



$$p(t) = \ell(t) + 3\mathbf{a}_1^3(t) - 3\mathbf{a}_2^3(t).$$

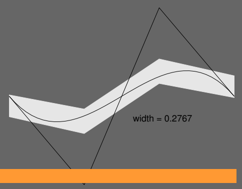
$\mathbf{a}_1^3$ is strictly convex, easy to bound above (and below)



$$p(t) \leq \overline{p}(t) = \ell(t) + 3\overline{\mathbf{a}_1^3}(t) - 3\underline{\mathbf{a}_2^3}(t)$$

$t =$	0	1/3	2/3	1
$a[3,3,-,1,..]$	-.0695214343	-.4398918047	-.3153515940	-.0087327217
$a[3,3,+,1,..]$	0	-.3703703704	-.2962962963	0

[4] of SLEFE Construction: Bézier

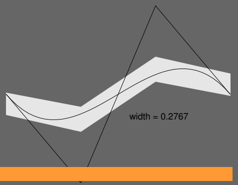


- Input Bézier piece of degree d: $x(u) := \sum_{k=0}^{d} \mathbf{b}_k(u) x_k$
- $\mathbf{h}_\mu^m$ hat functions with break points at $\frac{\mu}{m}$, $\mu = 0, \ldots, m$

$$\overline{x}(t) := \sum_{\mu=0}^{m} \tilde{x}_\mu \mathbf{h}_\mu^m(t), \quad \text{where} \quad [\tilde{x}_0, \ldots, \tilde{x}_m] := \text{slefe}([x_0, x_1, \ldots, x_d], m, +1)$$

$$\underline{x}(t) := \sum_{\mu=0}^{m} \underset{\sim}{x}_\mu \mathbf{h}_\mu^m(t), \quad \text{where} \quad [\underset{\sim}{x}_0, \ldots, \underset{\sim}{x}_m] := \text{slefe}([x_0, x_1, \ldots, x_d], m, -1).$$

[4] of SLEFE Construction: Bézier



$$\overline{x}(t) := \sum_{\mu=0}^{m} \tilde{x}_\mu \mathbf{h}_\mu^m(t), \qquad \text{where } [\tilde{x}_0, \ldots, \tilde{x}_m] := \text{slefe}([x_0, x_1, \ldots, x_d], m, +1)$$

$$\text{slefe}([x_0, \ldots, x_d], m, sgn) := [q_0, \ldots, q_m]$$

$$q_\mu := x_0(1 - \frac{\mu}{m}) + x_d(\frac{\mu}{m}) + \sum_{\nu=1}^{d-1} \mathcal{F}_\nu x \; a_{\nu,\mu}$$

$$\mathcal{F}_\nu x := x_{\nu-1} - 2x_\nu + x_{\nu+1},$$

$$a_{\nu,\mu} := a[d, m, sign(\mathcal{F}_\nu x) \times sgn, \nu, \mu].$$

[4] of SLEFE Construction: Bézier

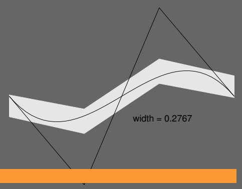
For degree d and m segments $1 \leq \nu \leq d, \quad 1 \leq \mu \leq m$ and $sgn \in \{-1, +1\}$
$a[d, m,\ sgn,\ \nu, \mu]$ is a table of numbers

[to be obtained, for example from the SubLiME web page]

For example,

$t =$	0	1/3	2/3	1
$a[3, 3, -, 1, ..]$	-.0695214343	-.4398918047	-.3153515940	-.0087327217
$a[3, 3, +, 1, ..]$	0	-.3703703704	-.2962962963	0

[4] of SLEFE Construction: Bézier



Input:

x_k,

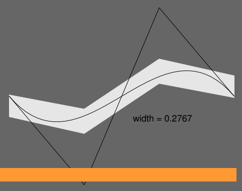
$a[d, m,\ sgn,\ \nu, \mu]$,

slefe$([x_0, \ldots, x_d], m, sgn)$. 2d(m+1) ops

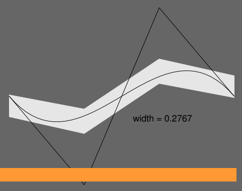
Output:

$$\overline{x} \quad \geq \quad x \quad \geq \quad \underline{x}$$

SLEFE properties



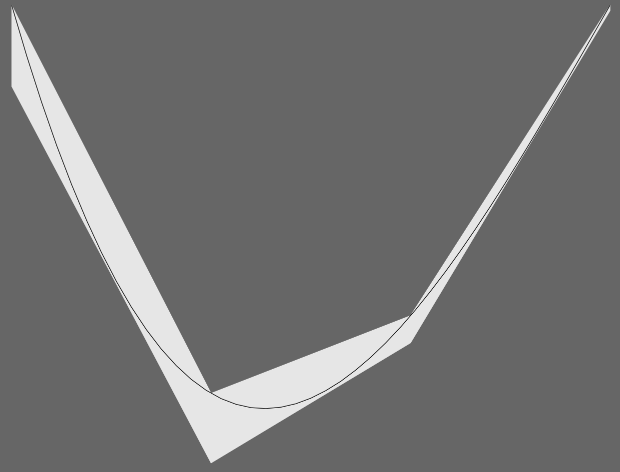
SLEFE properties



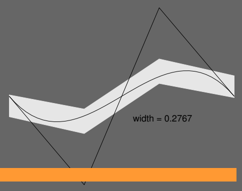
Near Optimality: the *width*

$$w_\mu := \overline{x}(\frac{\mu}{m}) - \underline{x}(\frac{\mu}{m}) = \sum_\nu (a[d,m,+,\nu,\mu] - a[d,m,-,\nu,\mu])|\Delta_\nu^2 x|.$$

is close to minimal in the *recursively applied L^∞ norm*.



SLEFE properties



Near Optimality: the *width*

$$w_\mu := \overline{x}\left(\frac{\mu}{m}\right) - \underline{x}\left(\frac{\mu}{m}\right) = \sum_\nu (a[d,m,+,\nu,\mu] - a[d,m,-,\nu,\mu])|\Delta_\nu^2 x|.$$

Nonlinear problem – cannot expect linear SLEFE construction to be optimal!

(1) optimal for degree $d = 1, 2$
(2) $d = 3$ no inflection, $m = 3$ segments:
optimal enclosure can be computed (nontrivial); ratio of minimal width to SLEFE width is < 1.07.
(3) $d = 4$ no inflection, $m = 4$ segments: < 1.04

General theorem still missing! (It depends on B, U, H, $\mathcal{F}$)

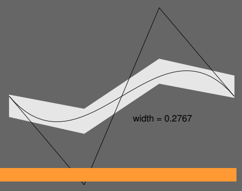
Difficulty: useful characterization of optimal enclosure (note piecewise!)

SLEFE properties

[1] Near Optimality: $w_\mu := \sum_\nu (a[d,m,+,\nu,\mu] - a[d,m,-,\nu,\mu])|\Delta_\nu^2 x|$

[2] *Invariance under addition of linear terms*

SLEFE properties

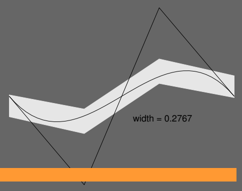


[1] Near Optimality: $w_\mu := \sum_\nu (a[d,m,+,\nu,\mu] - a[d,m,-,\nu,\mu])|\Delta_\nu^2 x|$

[2] Translation invariance

[3] *Refinement:* *Increase segments* or *Subdivide*.

SLEFE properties



[1] Near Optimality: $w_\mu := \sum_\nu (a[d,m,+,\nu,\mu] - a[d,m,-,\nu,\mu])|\Delta_\nu^2 x|$

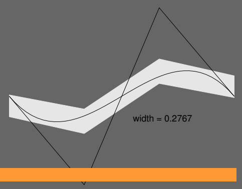
[2] Translation invariance

[3] *Refinement:* *Increase segments* or *Subdivide*.

In the limit, w_μ shrinks by $1/4$. ($\max_j |\Delta_\nu^2 x|$ shrinks to $\leq 1/4$ its size).

For specific $a[..]$ (Bézier), every w_μ shrinks by at least $\frac{3}{8}$ for $d = 2, 3, 4$.

SLEFE properties



[1] Near Optimality: $w_\mu := \sum_\nu (a[d,m,+,\nu,\mu] - a[d,m,-,\nu,\mu])|\Delta_\nu^2 x|$

[2] Translation invariance

[3] *Refinement:* *Increase segments* or *Subdivide*.

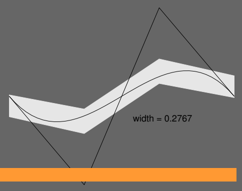
If the second differences are replaced by

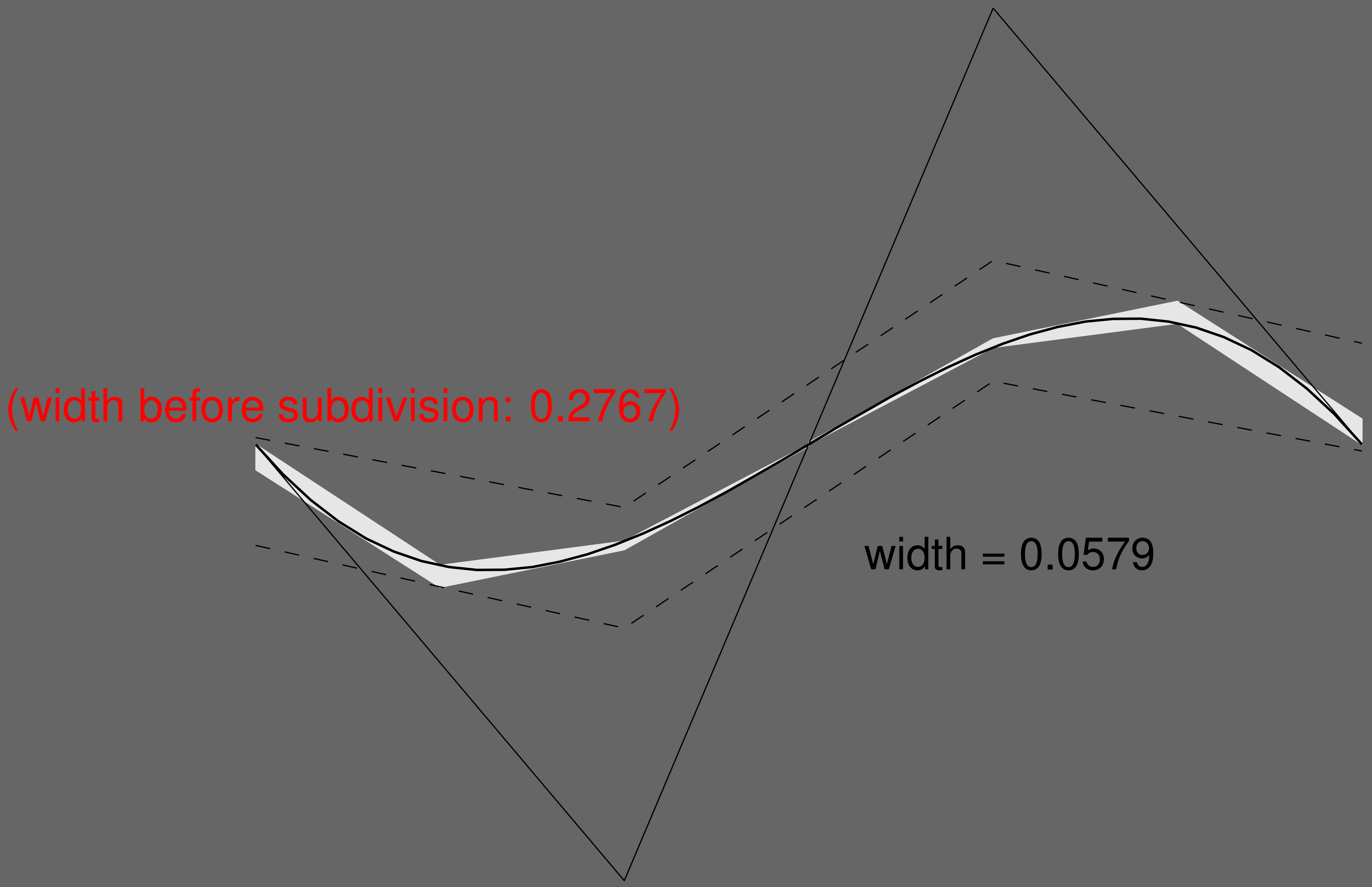
$$\sum_{i=0}^{d} \Delta_i^\nu(x), \qquad \Delta_i^\nu \text{ is } \nu\text{th difference applied to } x_i, \ldots, x_{i+\nu+1}.$$

then *every* w_μ shrinks by $\frac{1}{2}$ at every step.

(crucial estimate of proof: $\sum_{r=0}^{n} \binom{r}{c} / 2^r < 2$ for r, c, n nonnegative integers)

Subdivision





Are SLEFEs nested under subdivision?

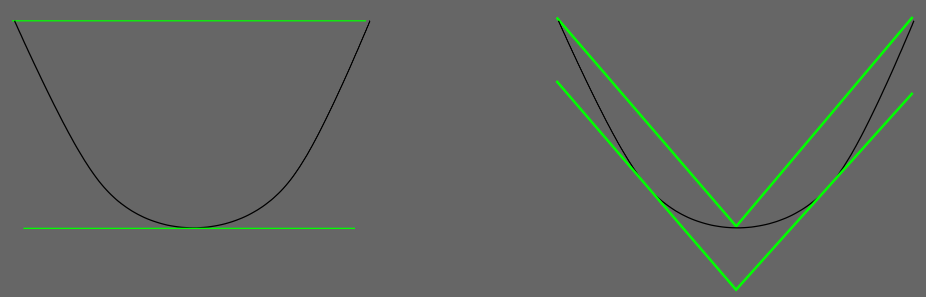
SLEFE properties

Near Optimality

Translation invariance

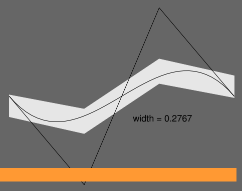
Refinement: Increase segments or Subdivide

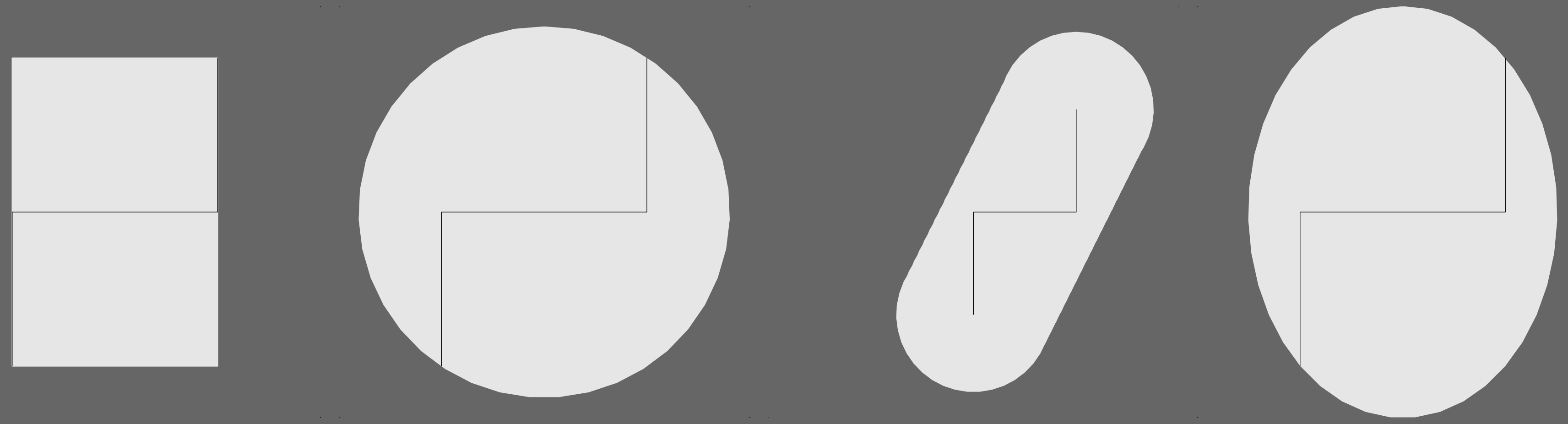
Not nested:



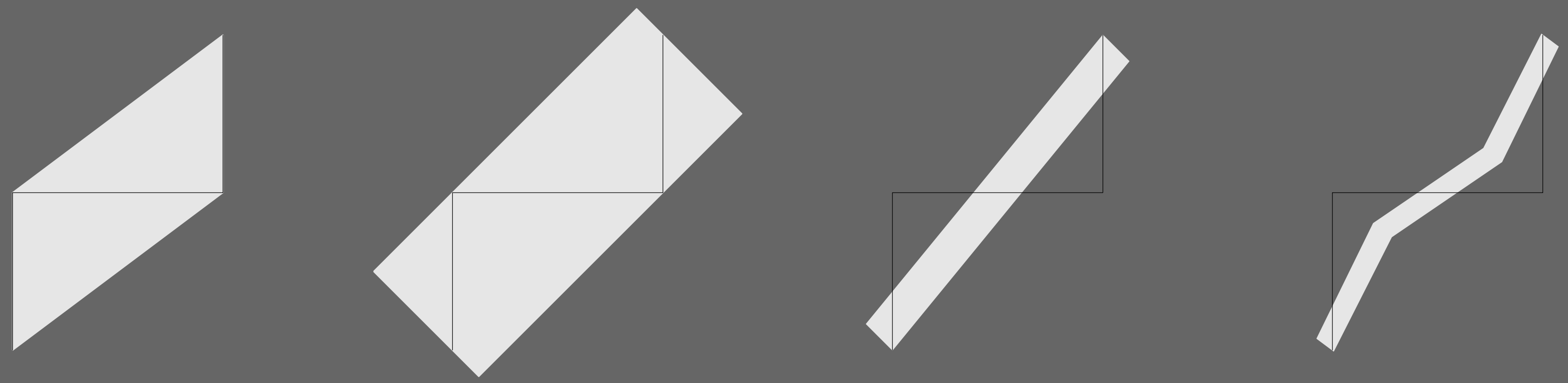
. How good are SLEFEs compared to other bounding constructs?

Bounding Constructs in Comparison



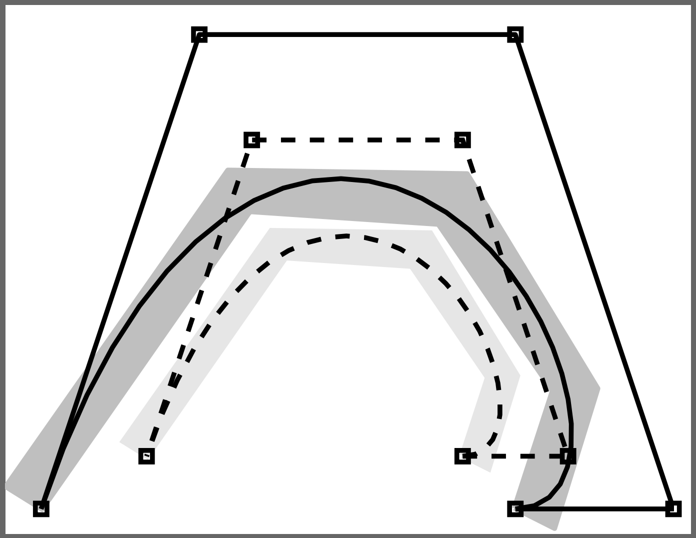


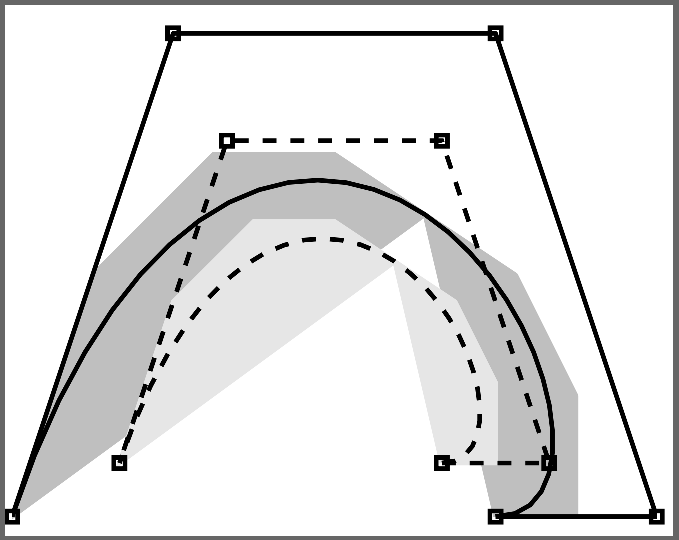
axis-aligned box, bounding circle, Filip bound bounding ellipse



convex hull, oriented bounding box, ‘fat arc’, *3-piece slefe*.

Intersection testing

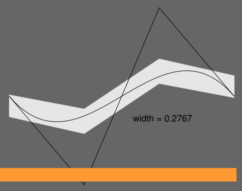




SLEFE (separation)

subdivided convex hull (no separation)

SLEFE construction: reprise



Given: B space of functions to be enclosed.

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H enclosure functions.

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3) Compute $\underline{\mathbf{a} - L\mathbf{a}} \in H$ and $\overline{\mathbf{a} - L\mathbf{a}} \in H$ (lower and upper bound)

4) Compute $\mathcal{F}x$ and assemble $\underline{x}$ and $\overline{x}$.

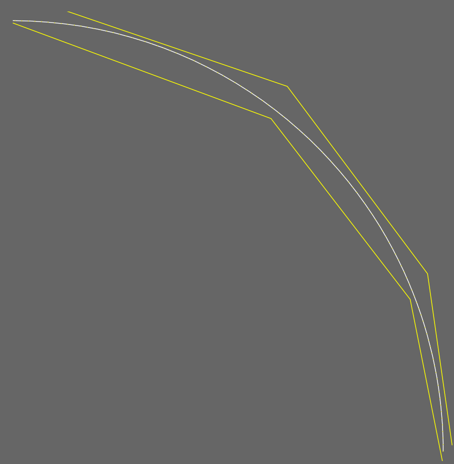
Note:
(0),(1),(2),(3) are offline
(4) is cheap

more SLEFEs

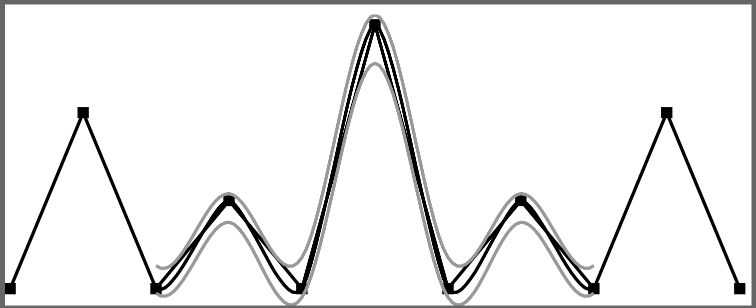
How widely is the construction applicable?

more SLEFEs

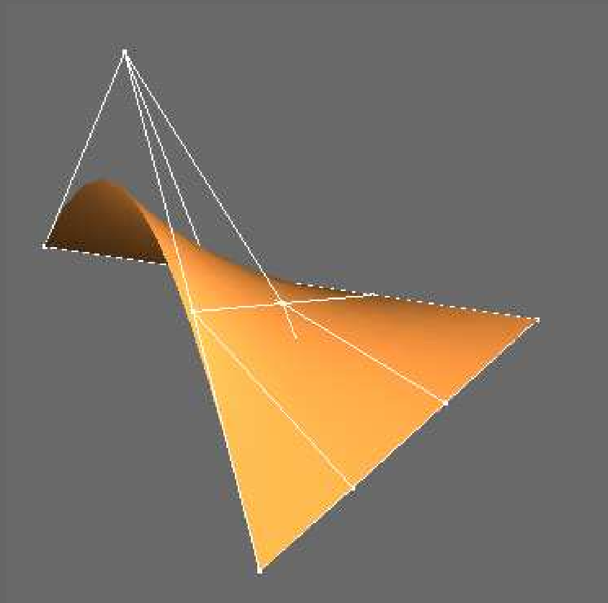
Rational functions:
Compute SLEFE of components + SLEFE for rational linear function

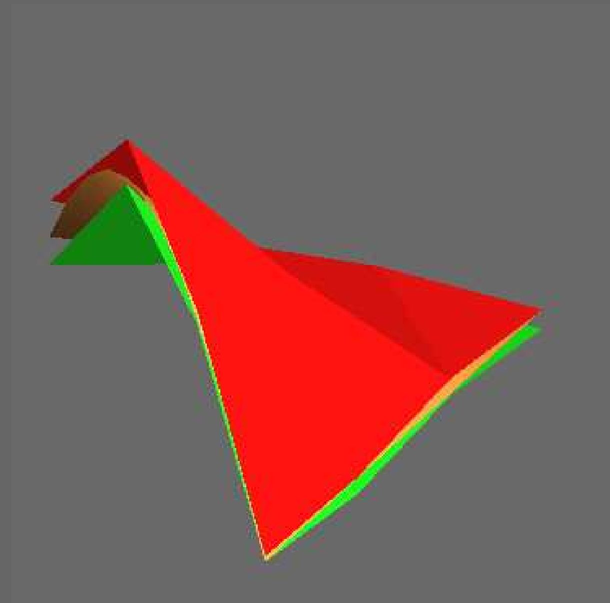


$B =$ 4-point scheme, $H =$ cubic splines ($\mathcal{F} =$ 4th differences):



3-sided patches





$B =$ total degree Bézier

U = unit triangle

$H =$ bivariate hat functions

$$\mathcal{F}_\nu x := \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Tensor-product SLEFEs

$$x(s,t) := \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} x_{ij} \mathbf{b}_j^{d_2}(t) \mathbf{b}_i^{d_1}(s). \qquad \mathbf{b}_k^d(u) := \frac{d!}{(d-k)!k!}(1-u)^{d-k}u^k$$

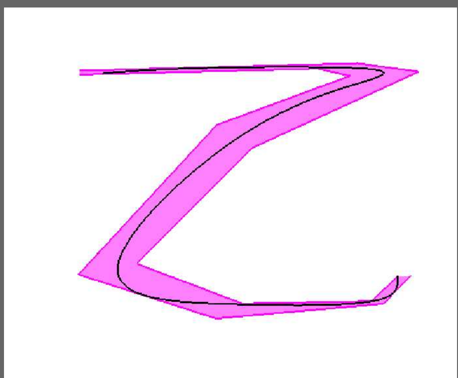
for $i = 0, \ldots, d_1, \quad [\tilde{x}_{i0}, \tilde{x}_{i1}, \ldots, \tilde{x}_{im_2}] :=$ slefe$([x_{i0}, x_{i1}, \ldots, x_{id_2}], m_2, +1)$

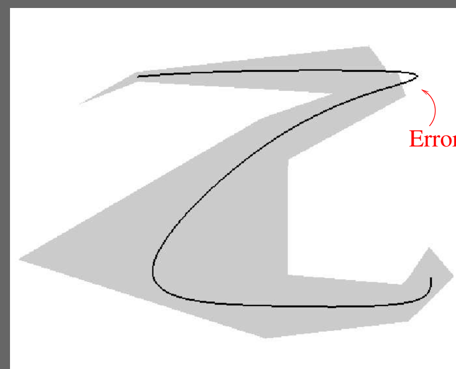
for $j = 0, \ldots, m_2, \quad [\tilde{\tilde{x}}_{0j}, \tilde{\tilde{x}}_{1j}, \ldots, \tilde{\tilde{x}}_{m_1 j}] :=$ slefe$([\tilde{x}_{i0}, \tilde{x}_{i1}, \ldots, \tilde{x}_{im_2}], m_1, +1)$.

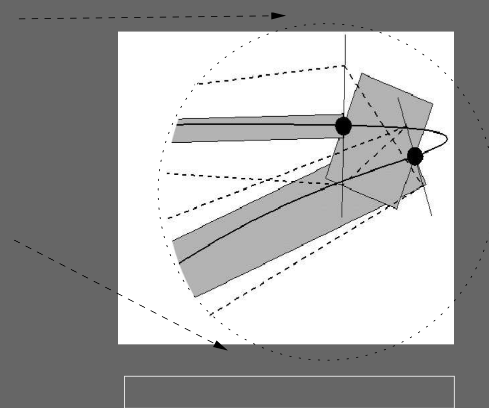
$$\overline{x}(s,t) := \sum_{j=0}^{m_1} \sum_{i=0}^{m_2} \tilde{\tilde{x}}_{ij} \mathbf{h}_i^{m_2}(s) \mathbf{h}_j^{m_1}(t).$$

Spline Curves

nontrivial!



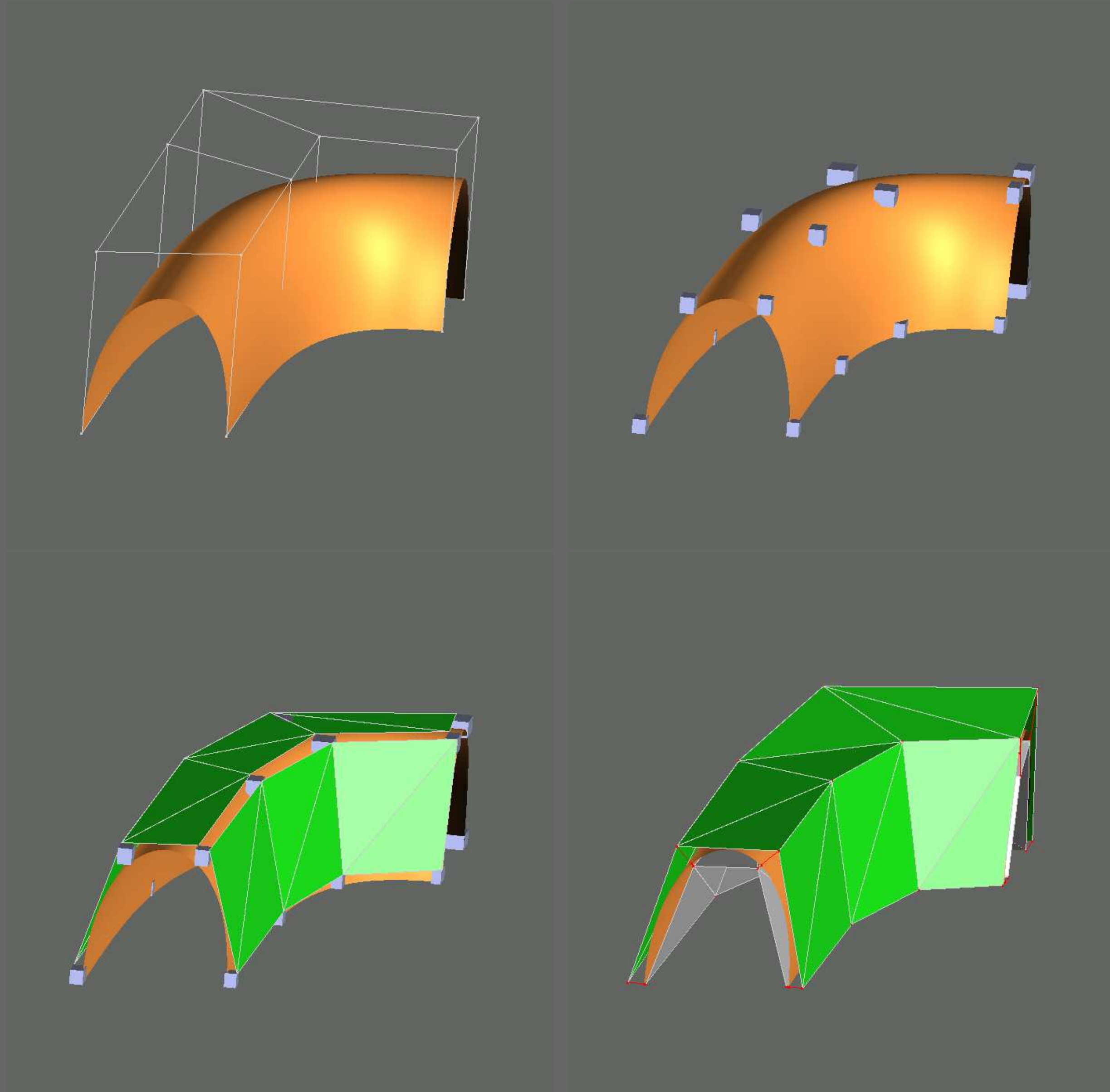




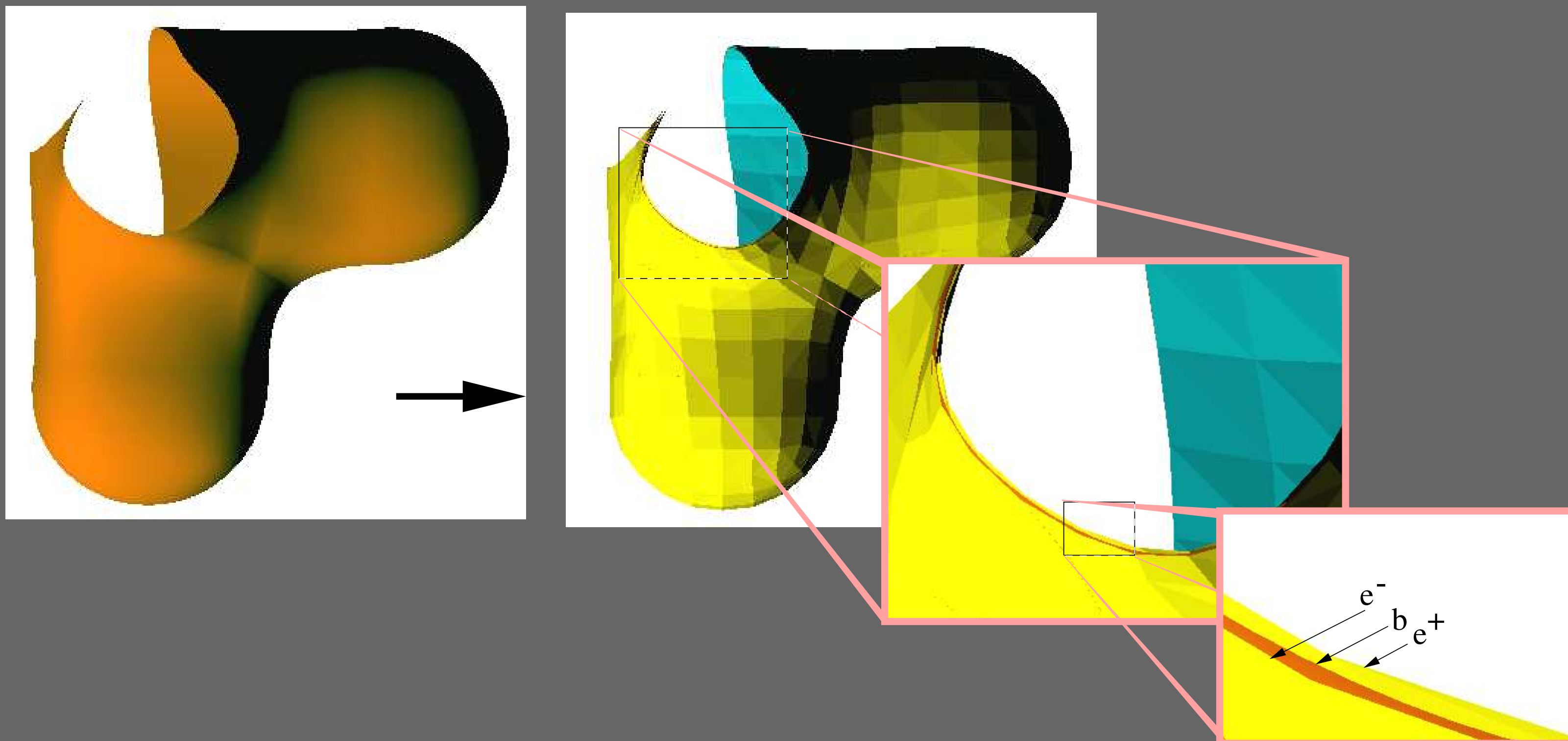
SLEVE (correct)
(P & Wu 2003)

alternative construction in the literature

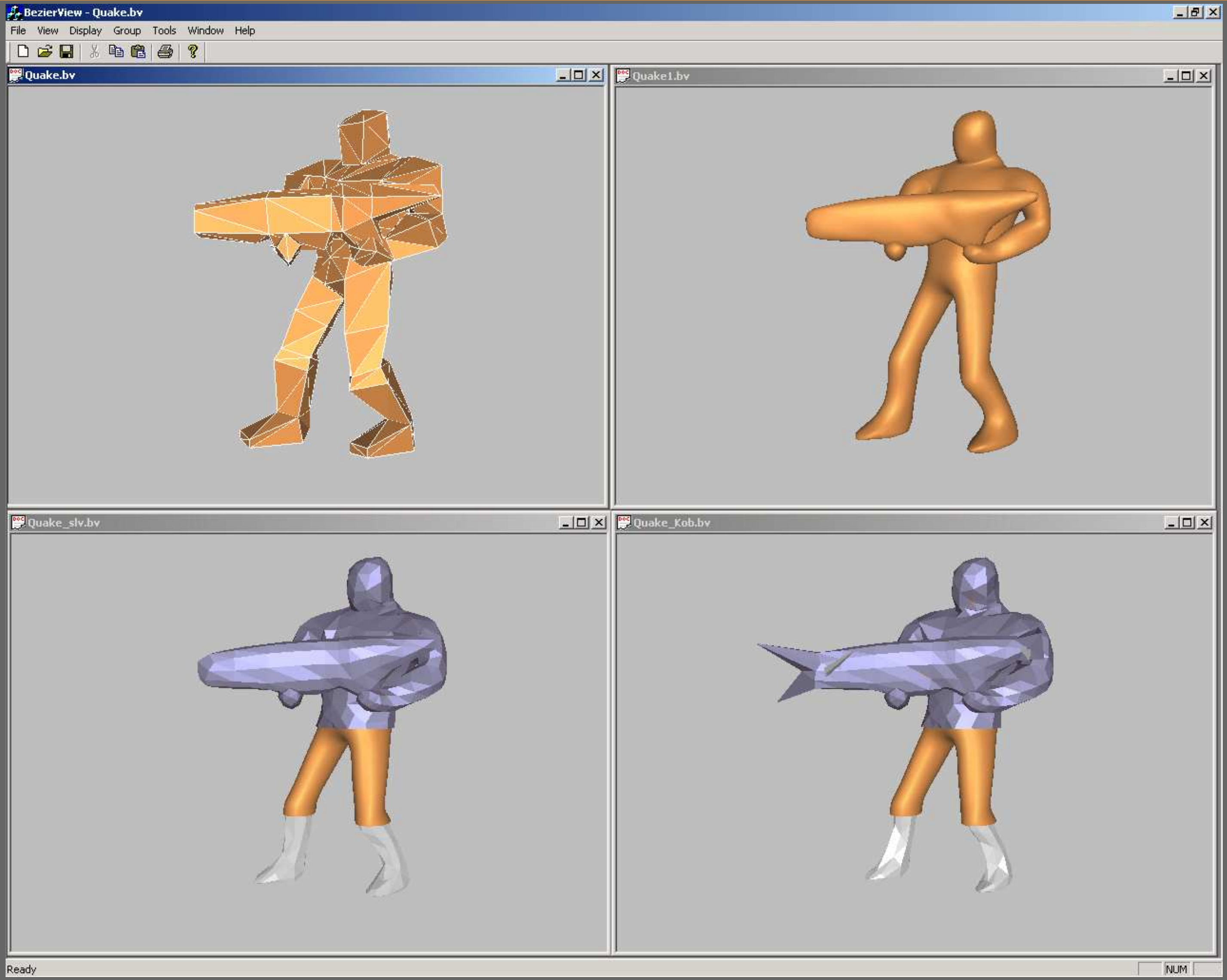
Spline surfaces



Spline surfaces

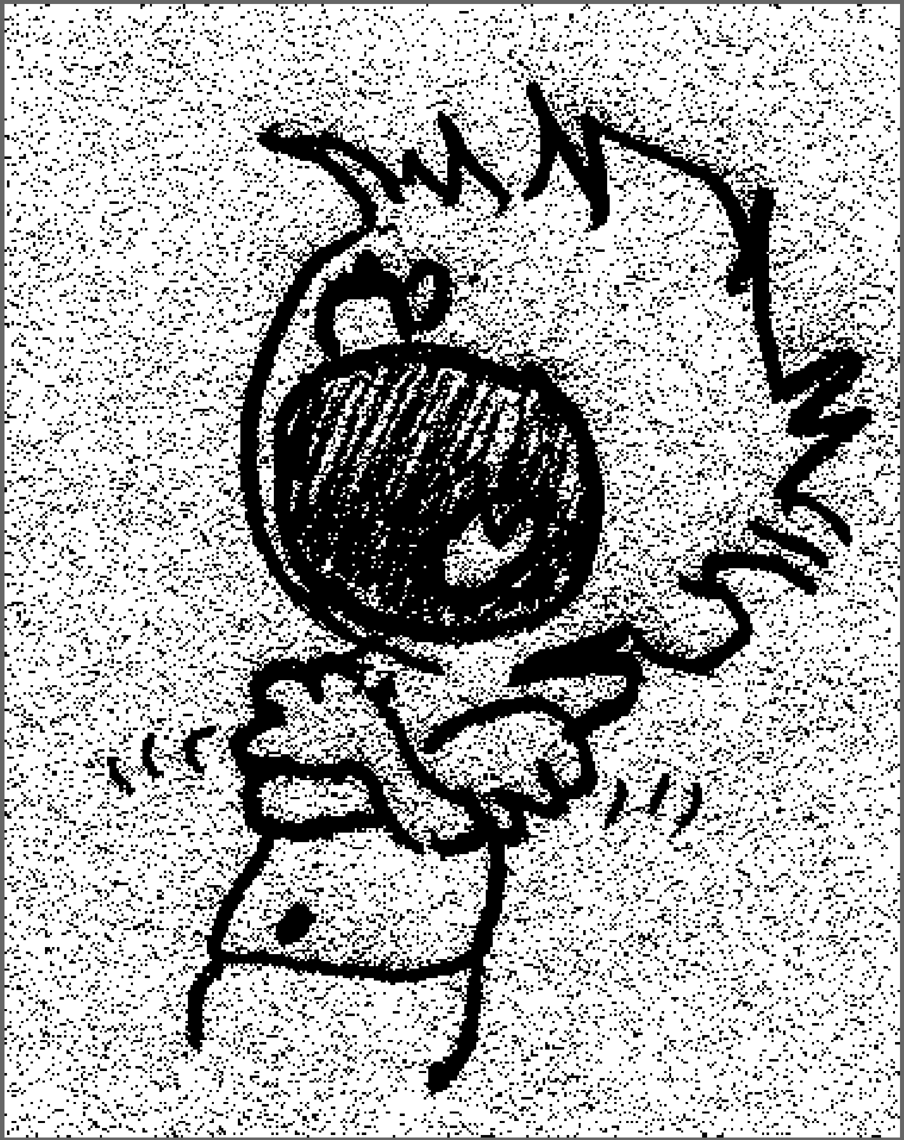


Subdivision schemes



. SLEFE alternative (incorrect!)

Life before SLEFEs



Life with SLEFEs