Mid-Structures Linking Curved and Piecewise Linear Geometry

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Abstract. Bézier or b-spline control meshes are quintessential CAGD tools because they link piecewise linear and curved geometry by providing a linear, refinable approximation that exaggerates features and is, up to reparametrization, in 1-1 correspondence with the curved geometry. However, for a given budget of line segments, Bézier and b-spline control meshes are usually far from the 'nearest' piecewise linear approximant to the curved geometry.

Subdividable Linear Efficient Function Enclosures, short slefes (pronounced like sleeves), aim at sandwiching the curved geometry as tightly as possible. This paper illustrates how to derive slefes, lists the literature on slefes, discusses slefes for rational functions and tensor-products and analyzes the improvement of slefes under refinement. The average of the upper and lower slefe bounds is called mid-structure. Mid-structures come close to being the 'nearest' piecewise linear approximant while retaining the 1-1 correspondence and the computational efficiency of control meshes.

§1. Introduction

What do interference testing of subdivision limit surfaces (Figure 1, (*left*)) and the bend-minimizing routing of spline curves between obstacles (Figure 1, (*right*)) have in common? Both tasks can be performed by enclosing the curved geometry by piecewise linear geometry. This, in turn, can be computed efficiently by computing a piecewise linear pair, $\overline{f}, \underline{f}$, of upper and lower bounds that tightly sandwich a given *function* f on a domain U: $\overline{f} \geq f \geq \underline{f}$. We call such a pair a *subdividable linear efficient function enclosure* of f, short slefe (pronounced like shirt sleeve) of f. That is, to compute a tight bound, we take advantage of the known parametrization of the object.

Slefe constructions for specific representations, splines in one and more variables, in Bézier and B-spline form and even for interpolatory subdivision

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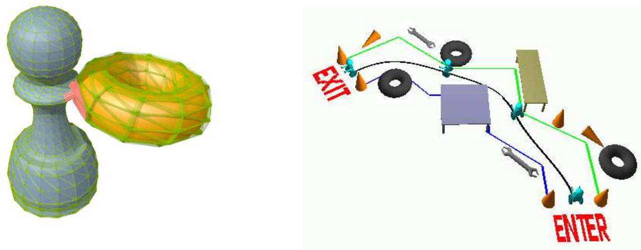

Fig. 1. Two applications of slefes. (*left*) Efficient and accurate intersection detection of subdivision limit surfaces [15]. (*right*) A robot navigates free space following an approximately curvature-minimizing spline path [8].

schemes have been reported in [6, 7, 10]. However, as pointed out in the survey [10], the underlying principle is the same. Each construction only differs by the choice of functionals and pre-computed best approximations that depend on those functionals. The quality of slefes is judged by their *width* $\mathsf{w}(f, U) := \overline{f} - \underline{f}$, with the arguments restricted to U. In simple cases, slefes have been compared to the best possible, i.e. narrowest, enclosure by a pair of piecewise linear bounds lines with the same break-abscissae [11]. The survey [10] juxtaposes slefes with eight other bounding constructs. The comparison is particularly in favor of slefes if the curve or surface is not close to linear, as in Figure 2. To form a consistent inner and outer hull of an object, slefes can be pieced together. This is explained, for different scenarios, in [12] and [15]. Fitting slefes between prescribed upper and a lower bounds, as in Figure 1 (*right*) , addresses a problem similar to near-interpolation [2].

The present paper sheds light on two additional aspects of slefes: refinement and mid-structures. Section 2 derives a particular slefe and then generalizes it to motivate the constructions and analysis of the following sections. Section 3 discusses how subdivision improves slefes and the width of the slefe changes under subdivision of f. Section 4 shows that slefes are easily extended to rational rep-

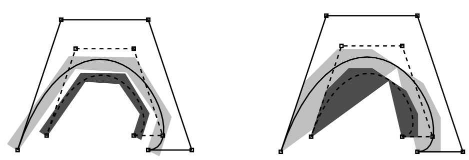

Fig. 2. (*left*) Separated slefes certify non-intersection. In contrast, the convex hulls of the curves overlap even after one subdivision (*right*) .

resentations even though rational representations lack a finite basis. Section 5 discusses the *mid*, $\underline{\overline{f}} := (\overline{f} + \underline{f})/2$, of a slefe, which serves as a good pointwise L^∞ approximant and plays an important role in efficient intersection testing.

§2. Example of a slefe

This section introduces slefes by means of a simple example. Consider the polynomial p of degree 3,

$$p(t) := -\mathsf{b}_1(t) + \mathsf{b}_2(t), \qquad \mathsf{b}_j := \binom{3}{j}(1-t)^{3-j}t^j,$$

in Bézier form $\sum_{j=0}^{3} c_j \mathsf{b}_j$, with coefficient sequence $(c_j)_{j=0,..,3} = (0,-1,1,0)$. In terms of its linear interpolant at $t=0$ and $t=1$, $\ell(t) := p(0)(1-t) + p(1)t$, and the new basis

$$\mathsf{a}_1(t) := -\frac{2}{3}\mathsf{b}_1(t) - \frac{1}{3}\mathsf{b}_2(t), \quad \mathsf{a}_2(t) := -\frac{1}{3}\mathsf{b}_1(t) - \frac{2}{3}\mathsf{b}_2(t),$$

we can rewrite the polynomial as

$$p = \ell + 3\mathsf{a}_1 - 3\mathsf{a}_2.$$

The function a_1 is strictly convex (see Figure 3 (*right*)) and is therefore easy to bound by a sequence of m connected line segments from above and from below. For example, for $m=3$, the four breakpoints of the

$$\text{piecewise linear upper bound function} \quad \overline{\mathsf{a}_1}^m(t), \quad t \in U := [0..1],$$

are $\mathsf{a}_1(j/3)$, $j = 0,1,2,3$. (If the number of line segments is evident or not important, we shorten the expression to $\overline{\mathsf{a}_1}$.) The four breakpoints of the piecewise linear lower bound function $\underline{\mathsf{a}_1}_3$ are obtained by parallel offsetting from the upper values until the segment touches the curve tangentially. Then we adjust, from the largest offset downwards, to hug the curve tightly. The result is stored in the following table (cf. Figure 3 (*right*)):

$t =$	0	1/3	2/3	1
$\overline{\mathsf{a}_1}^3$	0	-.3703703704	-.2962962963	0
$\underline{\mathsf{a}_1}_3$	-.0695214343	-.4398918047	-.3153515940	-.0087327217

Since $\mathsf{a}_2(1-t) = \mathsf{a}_1(t)$, the table for a_2 is mirror-symmetric. The point of this exercise is that, for $t \in U = [0..1]$,

$$p(t) \le \overline{p}(t) := \ell(t) - 3\underline{\mathsf{a}_1}_3(t) + 3\overline{\mathsf{a}_2}^3(t)$$

as illustrated in Figure 3 (*left*) . With the same tables, we have

$$p(t) \ge \underline{p}(t) := \ell(t) - 3\overline{\mathsf{a}_1}^3(t) + 3\underline{\mathsf{a}_2}_3(t).$$

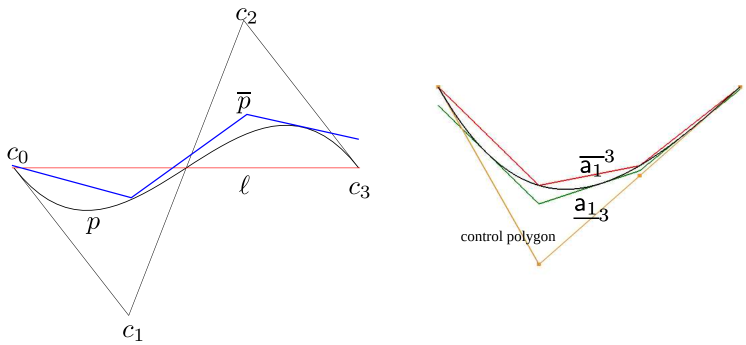

Fig. 3. (*left*) The function $p := -\frac{2}{3}\mathsf{b}_1 + \frac{1}{3}\mathsf{b}_2$, and its upper bound $\overline{p}$. (*right*) The lower and the upper bound $\underline{\mathsf{a}_1}_3$ and $\overline{\mathsf{a}_1}^3$ sandwiching the function a_1 with three segments.

Generalizations of the example.
The weights, $-3 = ((-1) - 2(1) + 0)$, $3 = (0 - 2(-1) + 1)$ are the result of evaluating second differences of the control points. That is, they are the result of applying two particular functionals F_ν to p, namely $F_1 p := c_0 - 2c_1 + c_2$ and $F_2 p := c_1 - 2c_2 + c_3$. It is then easy to check that the functions a_η are dual to the F_ν in the following sense.

Lemma 1. *For $\nu \in \{1, \dots, d-1\}$, define the functional $F_\nu(\sum_{k=0}^{d} c_k \mathsf{b}_k) := c_{k-1} - 2c_k + c_{k+1}$, and the scalar-valued functions*

$$\mathsf{a}_\nu^d := \frac{1}{\frac{\nu-1}{\nu} + \frac{d-\nu-1}{d-\nu} - 2} \Big(\sum_{k=0}^{\nu} \frac{k}{\nu} \mathsf{b}_k^d + \sum_{k=\nu+1}^{d} \frac{d-k}{d-\nu} \mathsf{b}_k^d \Big).$$

Then $F_\nu \mathsf{a}_\eta^d = \begin{cases} 1 & \text{if } \nu = \eta \\ 0 & \text{else.} \end{cases}$

Generally, for a polynomial p of degree d, (we drop the superscript d indicating the degree if the degree is evident or not important in the context)

$$p \le \overline{p} \qquad \overline{p} := \ell + \sum_{\nu=1}^{d-1} \max\{0, F_\nu p\}\, \overline{\mathsf{a}_\nu^d} + \sum_{\nu=1}^{d-1} \min\{0, F_\nu p\}\, \underline{\mathsf{a}_\nu^d}.$$

A lower bound $\underline{p}$ is obtained by exchanging the $\min$ with the $\max$ operators. The setup can be further generalized until it can be summarized in two short and abstract lemmas, Lemma 1 and 2 of [10].

The key challenge lies in generating once and for all, for each new representation, the tables $\mathbf{a}[\dots]$ that record the break points of the upper and lower bounds minimizing

$$\mathsf{w}(\mathsf{a}_\nu, U) := \overline{\mathsf{a}_\nu} - \underline{\mathsf{a}_\nu}.$$

Here, we record the breakpoint values of $\overline{\mathsf{a}_\nu} - \underline{\mathsf{a}_\nu}$ in a vector and minimize with respect to the L^∞ norm (the largest entry) as follows. For piecewise linear enclosures, the width is to be as small as possible where it is maximal. Having fixed the values at the pair of breakpoints where the width is maximal (zeroth and first breakpoint in Figure 3 (*right*)), the width at the remaining breakpoints is recursively minimized subject to matching the already fixed break point values.

By generating tight slefes for the a_ν, we expect to stay close to optimal when we compute a linear combination of the tight bounds and measure $\mathsf{w}(f, U) := \overline{f} - \underline{f}$. (To exactly minimize the width would imply that we solved a nonlinear problem by a single linear approximation, and should therefore not be expected.)

The upshot of all this is: if someone provides the tables $\mathbf{a}[\ldots]$ of the breakpoint values of $\overline{\mathsf{a}_\nu}$ and $\underline{\mathsf{a}_\nu}$ then an enclosure can be computed cheaply by the following algorithm.

An algorithm for computing the slefe of a polynomial in Bézier form

Let $f := \sum_{k=0}^d \mathsf{b}_k f_k$ be a linear combination of Bézier basis functions b_k with coefficients f_k. For (small) integers d, m, $1 \le \nu \le d$, $1 \le \mu \le m$ and $sgn \in \{-1, +1\}$, let

$$F_\nu f := f_{\nu-1} - 2f_\nu + f_{\nu+1}$$

$$q_\mu := (1 - \frac{\mu}{m}) f_0 + f_d(\frac{\mu}{m}) + \sum_{\nu=1}^{d-1} F_\nu f \; \mathbf{a}[d, m, sign(F_\nu f) \times sgn, \nu, \mu]$$

and $\mathbf{a}[d, m, sgn, \nu, \mu]$ a table of breakpoint values (available, say via [14]). Then

$$\mathsf{slefe}([f_0, \ldots, f_d], m, sgn) := [q_0, \ldots, q_m].$$

Let h_μ^m be the piecewise linear hat function with break points at $\frac{j}{m}$, $j = 0, \ldots, m$ that is 1 at $\frac{\mu}{m}$ and 0 at all other break points. Then the m-piecewise linear upper and lower component of the slefe are for $t \in [0..1]$,

$$\overline{f}(t) := \sum_{\mu=0}^m \tilde{f}_\mu \mathsf{h}_\mu^m(t), \quad \text{where } [\tilde{f}_0, \ldots, \tilde{f}_m] := \mathsf{slefe}([f_0, f_1, \ldots, f_d], m, +1)$$

$$\underline{f}(t) := \sum_{\mu=0}^m \underset{\sim}{f}_\mu \mathsf{h}_\mu^m(t), \quad \text{where } [\underset{\sim}{f}_0, \ldots, \underset{\sim}{f}_m] := \mathsf{slefe}([f_0, f_1, \ldots, f_d], m, -1).$$

Slefes of polynomials in tensor-product form

$$f(s,t) := \sum_{i=0}^{d_1}\sum_{j=0}^{d_2} f_{ij}\mathsf{b}_j^{d_2}(t)\mathsf{b}_i^{d_1}(s). \qquad \mathsf{b}_k^d(u) := \frac{d!}{(d-k)!k!}(1-u)^{d-k}u^k,$$

are easily generated on $[0..1]^2$ from the univariate slefe on $[0..1]$ as follows.

$$\text{for } i = 0,\ldots,\ d_1, \quad [\tilde f_{i0}, \tilde f_{i1}, \ldots, \tilde f_{im_2}] := \mathsf{slefe}([f_{i0}, f_{i1}, \ldots, f_{id_2}], m_2, +1)$$
$$\text{for } j = 0,\ldots,m_2, \quad [\tilde{\tilde f}_{0j}, \tilde{\tilde f}_{1j}, \ldots, \tilde{\tilde f}_{m_1 j}] := \mathsf{slefe}([\tilde f_{0j}, \tilde f_{1j}, \ldots, \tilde f_{d_2,j}], m_1, +1).$$

$$\overline{f}(s,t) := \sum_{j=0}^{m_1}\sum_{i=0}^{m_2} \tilde{\tilde f}_{ij}\mathsf{h}_i^{m_2}(s)\mathsf{h}_j^{m_1}(t).$$

An alternative slefe construction

Note that we could have chosen not only different functionals but also a different approximant ℓ that is anihilated by the functional. For example, since the functional is a second difference, we can choose the linear function ℓ_{12} that interpolates c_1 and c_2, write $p = \ell_{12} + (c_2 - 2c_1 + c_0)\mathsf{b}_0 + (c_3 - 2c_2 + c_1)\mathsf{b}_3$. If we then bound the basis functions b_0 and b_3, which happen to be convex for degree 3 and hence also easy to bound, be arrive at an alternative slefe construction for polynomial pieces of degree 3. The approach can be bootstrapped by subtracting from the input polynomial $p := \sum_{j=0}^{d} c_j\mathsf{b}_j$ the polynomial $p_{d-2} := \sum_{j=1}^{d-1} c_j\mathsf{b}_j + \tilde c_0\mathsf{b}_0 + \tilde c_d\mathsf{b}_d$ with $\tilde c_0$ and $\tilde c_d$ chosen so that p_{d-2} is of degree $d-2$. Then $p - p_2 = (c_0 - \tilde c_0)\mathsf{b}_0 + (c_d - \tilde c_d)\mathsf{b}_d$ can be bounded by bounding two convex functions and we can iterate by bounding p_{d-2} in degree-reduced form. This results in an expansion of p in terms of convex polynomial pieces on the interval $U = [0..1]$.

So why would we not just bound the original basis functions b_k^d to start with? The answer (for the particular functionals F_ν, namely second differences) is that adding any constant or linear function h would modify the width of the slefe and make it arbitrarily large! Specifi cally,

$$\sum_{j=0}^{d}(c_j + h)(\overline{\mathsf{b}_j} - \underline{\mathsf{b}_j}) = \sum_{j=0}^{d} c_j(\overline{\mathsf{b}_j} - \underline{\mathsf{b}_j}) + h\underbrace{\sum_{j=0}^{d}(\overline{\mathsf{b}_j} - \underline{\mathsf{b}_j})}_{const}.$$

By contrast, if F_ν is a second difference, $F_\nu(h) = 0$ so that the slefe construction and width are unaffected by translation of the function.

Figure 4 points to more general applicability of the approach. It shows the slefe of a '3-sided patch', a patch in total degree bivariate B´ezier form with U the unit triangle. Here

$$\mathsf{b}_{\mathbf{k}}, \quad \mathbf{k} := (k_0, k_1, k_2) \in \mathbb{N}^3 \text{ and } k_0 + k_1 + k_2 = 4$$

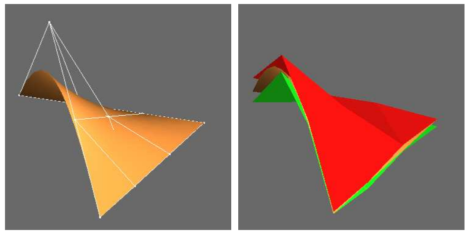

Fig. 4. (*left*) Bézier piece of total degree 4 with control structure. (*right*) The piece enclosed by its slefe.

are the basis functions of bivariate polynomials total degree 4 in Bézier form, h_μ the bivariate hat functions corresponding to a regular partition of the unit triangle and

$$F_\nu f := f_{\nu_0-2,\nu_1+1,\nu_2+1} - f_{\nu_0-1,\nu_1,\nu_2+1} - f_{\nu_0-1,\nu_1+1,\nu_2} + f_{\nu_0,\nu_1,\nu_2}$$

is a second difference and therefore anihilates linear components.

§3. Refinement and slefes

There are two alternative ways to refine the piecewise linear upper and lower bounds of a slefe. The first is to increase m, the number of segments when bounding a_ν above and below. This only mildly increases the runtime cost, but requires larger pretabulations. The second is to apply, at runtime, De Casteljau's algorithm to $p(t) := \sum_{k=0}^{d} c_k b_k(t)$, say at the midpoint $t = 1/2$ of the unit interval. This yields a left piece p_1 (and, similarly, a right piece p_2) when $t \in [0..1]$. The left piece represents p on $[0..\frac{1}{2}]$ with coefficients $S_d\mathbf{c}$ where $\mathbf{c}$ is the vector of coefficients of p and S_d is a matrix of size $d+1 \times d+1$,

$$S_d := (\binom{r}{q}/2^r)_{r,q\in\{0,\dots,d\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & \dots & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{2^d} & \dots & \dots & \dots & \dots & \frac{1}{2^d} \end{bmatrix}.$$

Figure 5 illustrates why good slefes should not be nested under refinement, i.e.

the optimal slefe after refinement should not generally fit
inside the optimal slefe before refinement.

By definition, the intersection of the slefes at different levels of refinement is again a piecewise linear enclosure.

Often, we want to guarantee a maximal width everywhere. It is difficult to estimate the number of subdivisions necessary, unless we have some fixed constant less than 1 of decay of the pointwise widths. So, an important question is, whether the width at an existing breakpoint can increase, or a new break point has a width that exceeds that of its neighboring old breakpoints.

To answer this question, we recall that the width at breakpoint $\frac{\mu}{m}$ is

$$w(p, \frac{\mu}{m}) := \overline{p}(\frac{\mu}{m}) - \underline{p}(\frac{\mu}{m}) = \sum_{\nu=1}^{d-1} (\overline{\mathsf{a}_\nu} - \underline{\mathsf{a}_\nu})(\frac{\mu}{m})|F_\nu p| =: W(\mu, :)\mathbf{F}(p).$$

Here, $W(\mu, :)$ is row μ of the matrix of widths of the functions a_ν at $\frac{\mu}{m}$,

$$W := \left(w_\nu(\frac{\mu}{m}) \right)_{\mu=0..m,\nu=1..d-1}, \qquad w_\nu := \overline{\mathsf{a}_\nu} - \underline{\mathsf{a}_\nu},$$

and $\mathbf{F}(p) := \left(|F_\nu(p)| \right)_{\nu=1..d-1}$ is the vector of absolute second differences of p.

Lemma 2. *If $w(f;\sigma) := W\mathbf{F}(f)$ is the vector of widths after σ subdivision steps then*

$$w(f;\sigma+1) \le WS_{d-2}\frac{1}{4}\mathbf{F}(f).$$

Proof: Let Δ be the $d \times d+1$ matrix that maps the vector of coefficients to their first differences. Due to the halving of the abscissae distances, $S_{d-1}\Delta = 2\Delta S_d$ for the subdivision matrix of the differences as elaborated by

$$\begin{array}{c} \ldots 024 \ldots \overset{S_d}{\to} \ldots 01234 \ldots \overset{\Delta}{\to} \ldots 111 \ldots \\ \ldots 024 \ldots \overset{\Delta}{\to} \ldots 222 \ldots \overset{S_{d-1}}{\to} \ldots 222 \ldots \end{array}$$

Therefore, if F is the matrix whose νth row represents the νth second difference then $FS_d = \Delta\Delta S_d = S_{d-2}F/4$ and (note the absolute values, applied in last when forming $|F(S_d(\mathbf{c}))|$ with $\mathbf{c}$ the coefficients of f):

$$\mathbf{F}S_d = \text{abs}\Delta\Delta S_d = \text{abs}S_{d-2}\frac{1}{4}\Delta\Delta \le \frac{1}{4}S_{d-2}\text{abs}\Delta\Delta = \frac{1}{4}S_{d-2}\mathbf{F}.$$

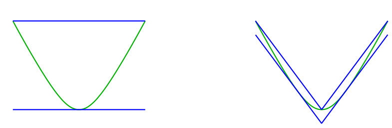

Fig. 5. Good enclosures are not nested. Refinement from 1 to 2 segments. The optimal slefe on the right does not fit inside the optimal slefe on the left.

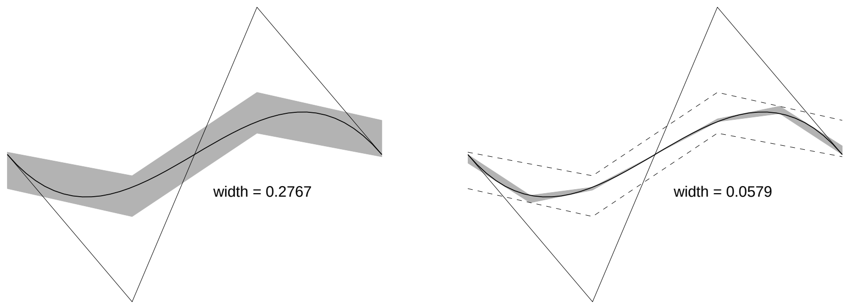

Fig. 6. (*left*) A cubic B´ezier segment with coeffi cients $0, -1, 1, 0$. The control polygon exaggerates the curve far more than the *grey* 3-piece slefe. (*right*) After one subdivision at the midpoint, the width of the slefe (*grey*) is roughly 1/4th of the width of the unsubdivided slefe (*dashed*).

The claim follows from $w(f; \sigma+1) = W\mathbf{F}S_d(f)$ □

Observation 3.1: If all F_ν are equal then $S\mathbf{F} = \mathbf{F}$ and all widths decrease by $\frac{1}{4}$. This happens in the limit, when the second differences converge.

Observation 3.2: To show that widths can, in principle, increase locally under subdivision, let v be a row of W. If the W is not further specifi ed, we could have $v(1) = 0$, $v(j) = 1$ for $j \neq 1$ and $\mathbf{F}_j = 0$ for $j \neq 1$, $\mathbf{F}_1 = 1$. Then $w(f, \sigma) = 0$ but $w(f, \sigma+1) = 1/2 + 1/4 + \ldots + 1/2^d \neq 0$. That is, the ratio of widths after and before the subdivision step could be infi nite at a specifi c breakpoints rather than the hoped-for $1/4$.

Observation 3.3: For the algorithm stated in Section 2, the entries of $\underline{\overline{a}}$ have their largest entry along the diagonal and the widths are all guaranteed decrease by at least $3/8$ for $d = 2, 3, 4, 5$.

We now show that, if the second differences are replaced by the sum of νth differences then *every* $v(i)$ shrinks by at least $1/2$ at *every* subdivision step regardless of the tightness of the estimates in tables $\mathbf{a}[\ldots]$. To prove the result, we fi rst estimate the column sums of S_d.

Lemma 3. *Each row of* S_d *sums to 1. The sum of all elements in each column* q *each column suq,* $s(d, q) := \sum_{r=0}^{d} \binom{r}{q}/2^r$*, is strictly bounded above by 2.*

Proof:

$$\begin{aligned} s(d,q) &= \frac{1}{2}s(d-1,q-1) + \frac{1}{2}s(d-1,q) \\ &= \frac{1}{2}s(d-1,q-1) + \frac{1}{2}[\frac{1}{2}s(d-2,q-1) + \frac{1}{2}s(d-2,q)] \\ &= \sum_{j=1}^{d-q} s(d-j,q-1)/2^j \text{ since } s(d-k,q) = 0 \text{ for } d-k < q. \end{aligned}$$

We observe $s(d,0) < 2$ for all d and use this as induction start from which $s(d,q) < 2$ follows as claimed. □

Lemma 4. *For $\nu = 2,\ldots,d$ define*

$$F_\nu(f) := \sum_{j=0}^{d+1-\nu} \Delta_j^\nu(f), \qquad \Delta_j^\nu \text{ is } \nu\text{th difference applied to } f_j,\ldots,f_{\nu+j+1}.$$

Let $f(t) := \sum_k f_k \mathsf{b}_k(t) = \sum_{\nu=1}^{d-1} F_\nu(f)\mathsf{a}_\nu(t)$, where each a_ν is a polynomial of degree d defined by $\mathsf{a}_\nu(0) = \mathsf{a}_\nu(1) = 0$, $F_\eta(\mathsf{a}_\nu) = 0$ if $\nu \neq \eta$ and $F_\nu(\mathsf{a}_\nu) = 1$. For the interval $U = [0..1]$, let $\underline{\mathsf{a}_\nu} \leq \mathsf{a}_\nu \leq \overline{\mathsf{a}_\nu}$, $\nu = 1..d-1$ be any *choice of lower and upper bounds. Then subdivision at $t = \frac{1}{2}$ reduces the width of the enclosure of p to less than $1/2$ the previous width.*

Proof: Let $\mathbf{1} := [1,\ldots,1]$ and Δ^ν the column vector of νth differences with jth entry Δ_j^ν so that $F_\nu = \mathbf{1}\Delta^\nu$. Then, as in the proof of Lemma 2,

$$\text{abs}F_\nu S_d = \text{abs}\mathbf{1}S_{d-\nu}\frac{1}{2^\nu}\Delta^\nu < \frac{2}{2^\nu}\text{abs}F_\nu,$$

where the inequality follows for the qth column of $S_{d-\nu}$ from $\mathbf{1}S_{d-\nu}(:,q) = s(d-\nu,q) < 2$ according to Lemma 3 since $\nu \geq 2$. □

The lemma gives a worst case estimate over all possible bounds W: regardless of how poorly or, and this is more important, how tightly we choose the enclosures, the width is guaranteed to halve everywhere.

§4. Slefes of Rational Functions

Rational functions are an example where we can not build a slefe directly as a linear combination of two-sided bounds on a finite family of functions since we do not have the finite basis. However, we can bound numerator and denominator of

$$r := \frac{p}{q} =: \frac{\sum p_k \mathsf{b}_k}{\sum q_k \mathsf{b}_k}.$$

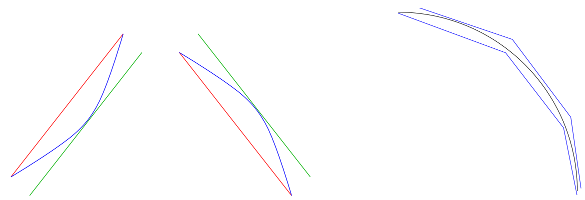

Fig. 7. (*left*) Enclosure of rational linear segment r_μ^+. (*right*) Quarter circle enclosed by its slefe.

separately and use elementary interval arithmetic. We must assume that $\underline{q} \neq 0$ on U, i.e. without loss of generality $\underline{q} > 0$. Then we compute $\overline{r}$ as follows. (The calculation of the lower enclosure is analogous). Let $\overline{p}_\mu$ be the μth breakpoint value, $\mu = 1, \ldots, m$, of $\overline{p}$. On the interval $[\frac{\mu}{m}..\frac{\mu+1}{m}]$, r is bounded above by the rational linear function

$$r_\mu^+ := \frac{\overline{p}_\mu(1-\alpha) + \overline{p}_{\mu+1}\alpha}{\underline{q}_\mu(1-\alpha) + \underline{q}_{\mu+1}\alpha} \qquad \alpha \in [0..1].$$

We determine the linear interpolant l_μ^0 to r_μ^+ at the breakpoints, and its parallel offset l_μ^1 that just touches r_μ^+ tangentially Figure 7 (*left*) . Depending on the convexity or concavity of r_μ^+, either the endpoints of l_μ^0 or of l_μ^1 provide a linear upper bound on r on the interval. By taking the maximum of the endpoints of abutting segments, adjacent slefe segments join continuously. A result is shown in Figure 7 (*right*) .

§5. Mid-Structures

Bézier or b-spline control meshes provide a linear, refinable approximation that exaggerates features and is, up to reparametrization, in 1-1 correspondence with the curved geometry. However, for a given budget of line segments, Bézier and b-spline control meshes are usually very loose piecewise linear approximations to the curved geometry. This section derives and analyzes a mid-structure (mid-path, mid-patch, etc.) that comes close to being the 'nearest' piecewise (bi-)linear approximant while retaining the 1-1 correspondence and the computational efficiency of control meshes.

Definition 1. *The mid of* f *is defined as* $\underline{\overline{f}} := (\overline{f} + \underline{f})/2$.

With Lf mapping to the piecewise linear functions, e.g. the ℓ in Section 2 or, alternatively, the control polygon of f, $\underline{\overline{\mathbf{a}}}$ the $d+1 \times m+1$ matrix of mids of

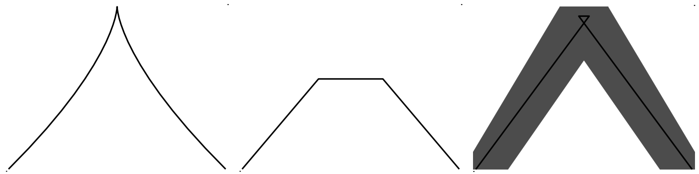

Fig. 8. A degree 3 curve (*left*) finely evaluated, (*middle*) approximated by sampling at four points, (*right*) approximated by a 3-segment mid-path.

the basis functions a_ν, and h the vector of hat functions h^m_μ, we can rewrite

$$\begin{aligned}\overline{\underline{f}}(t) &:= (\overline{f}(t)+\underline{f}(t))/2 = \frac{1}{2}\Big(\sum_{\mu=0}^{m}\tilde{f}_\mu \mathsf{h}^m_\mu(t) + \sum_{\mu=0}^{m}\underset{\sim}{f}_\mu \mathsf{h}^m_\mu(t)\Big)\\ &= f_0(1-t)+f_d t+\sum_{\mu=1}^{m}\sum_{\nu=1}^{d-1}F_\nu(f)\,\frac{\mathbf{a}[d,m,+,\nu,\mu]+\mathbf{a}[d,m,-,\nu,\mu]}{2}\mathsf{h}^m_\mu(t)\\ &= Lf(t)+F(f)\ \cdot\overline{\underline{\mathsf{a}}}\ \cdot\mathsf{h}(t).\end{aligned}$$

Observation 5.1: The mid $\overline{\underline{\mathbf{x}}} := (\overline{\mathbf{x}}+\underline{\mathbf{x}})/2$ is well-defined for a vector-valued curve or surface $\mathbf{x} := (x,y,z)$.

Observation 5.2: The boundary of a spline in piecewise Bézier form is, for example, the endpoint of a curve segment or the space curve corresponding to an edge for a patch in $\mathbb{R}^3$. Along such an edge, the mid-structure is computed from that boundary only. Therefore, mid-structures join continuously if their patches abut continuously. For example, we define the *mid-path* $\overline{\underline{f}}$ of f as the m-piece linear function with values

$$\overline{\underline{f}}\Big(\frac{\mu}{m}\Big) := \begin{cases}\frac{1}{2}(\overline{f}^m+\underline{f}_m)(\frac{\mu}{m}) & \text{if } 0<\mu<m,\\ f_\mu & \text{if } \mu=0 \text{ or } \mu=m.\end{cases}$$

The choice for $\mu = 0$ and $\mu = m$ guarantees that mid-paths of continuously joined Bézier pieces match up at their endpoints.

Observation 5.3: The distance between the polynomial f and the broken line $\overline{\underline{f}}$ on the interval $[\frac{\mu}{m}..\frac{\mu+1}{m}]$ is bounded by the linear average of the distances at the endpoints; and these distances are evidently bounded by

$$|f-\overline{\underline{f}}|\Big(\frac{\mu}{m}\Big) \le \frac{\epsilon_\mu}{2}(\overline{f}^m-\underline{f}_m)\Big(\frac{\mu}{m}\Big)$$

where $\epsilon_\mu = 2$ for $\mu = 0$ or $\mu = m$ and $\epsilon_\mu = 1$ otherwise. This makes $\overline{\underline{f}}$ an excellent max-norm approximation to the spline with a known maximal approximation distance. By contrast, naive linearization without further analysis, say triangulation by sampling, reapproximates without known error and typically with larger error between samples as illustrated in Figures 8 and 9.

Observation 5.4: (Midpath Control Structures) If the number of breakpoints equals the number of control points, for example if $m = d$ for a polynomial piece in B´ezier form, or $m = 1$ for each polynomial piece of a spline, then the matrix $\underline{\overline{\mathsf{a}}}$ is invertible for all (functionals and) tables encountered so far. Therefore, we can obtain f from $\underline{\overline{f}}$ by reversing the midpath coeffi cient computation,

$$\underline{\overline{f}} - Lf = \mathbf{F}(f) \cdot \underline{\overline{\mathsf{a}}}.$$

where Lf represents, for example for the linear interpolant ℓ in Section 2 and a is the vector of polynomials a_ν. Solving for $\mathbf{F}(f)$, we can reconstruct the function

$$f = Lf + \mathsf{a} \cdot\ \mathbf{F}(f) = Lf + \mathsf{a} \cdot (\underline{\overline{\mathsf{a}}})^{-1}(\underline{\overline{f}} - Lf)$$

from the known quantities Lf, $\underline{\overline{f}}$, $\underline{\overline{\mathsf{a}}}$ and a. That is, the mid-struct and control-polytope equivalently represent the (spline) function in different bases. This links piecewise linear with nonlinear spline geometry similar to control polygons, but with a closer spatial relationship. In particular, we can take the point of view that

any broken line can be interpreted as the $\underline{\overline{f}}$ of a spline f
of prescribed degree

with each control points associated with one break point. We check that $Lf = f$ is consistent.

The midpath for rational function can be inverted if we make additional assumptions on the convexity of the curve.

When deriving f from a broken line that lies in a plane, say the approximate level curve of an implicit function, it is good to know that f will stay in the same plane. More generally, the simple linear relation between f and $\underline{\overline{f}}$ implies the following.

Lemma 5. *$\underline{\overline{f}}$ and f lie in the same least dimensional hyperplane if Lf does.*

During the talk, an interactive example was shown where the interval intersection of the slefe, rather than of the exact function, was computed on the fly, and the piecewise linear central curve was interpreted and inverted as mid-structure. Also interactive manipulation of a cubic spline curve by its mid was shown. One potential drawback of using the mid-structure for design is that f equi-oscillates about the mid-structure if the slefe is effi cient, because then, $\underline{\overline{f}}$ is a (near-)optimal approximant in the recursive max-norm. More can be found in the master's thesis [3].

Why would we not just compute a best L^∞ approximant by Chebyshev economization? Chebyshev economization only generates optimal approximation from to polynomials of degree d from polynomials of degree $d-1$. Moreover, just like the standard Remez algorithm ([1] Section 6.1), it does not generate continuous *piecewise* approximations. Finally, neither approach yields the desirable one-sided approximation.

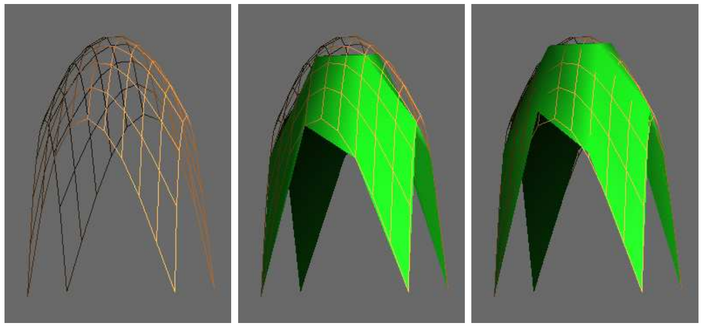

Fig. 9. A bi-quadratic Bézier patch (*left*) finely evaluated, (*middle*) approximated by sampling, (*right*) approximated by a mid-patch.

§6. Open Problems

Although there is by now a lot of empirical evidence that slefes a close to optimal in their width, it would be nice to *exactly* quantify how much we loose by switching from a hard nonlinear max-norm approximation problem to using the simple slefe construction. The difficulty lies in deriving the best approximation (if this were simple, we would indeed not need slefes) and determining the worst case.

While [15] indicates that slefes do a good job when used inside a collision detection hierarchy, the jury is still out as to whether it will be better than other methods at robustly finding *all roots* within some box U, say of a multivariate polynomial. Experiments with univariate polynomials, using a framework generously provided by Casciola and Fabbri of the University of Bologna, Italy, show that slefe-based root finding is on par with the best, Bézier clipping [13]. The hope is that the tighter bounds will pay of in the first steps of multivariate root finding.

The invertibility of the mid-structure opens up the possibility of parametrizing level sets approximately with a known error. Here one computes the interval intersection of the slefe, rather than the exact function, and uses the middle curve or surface of the interval intersection as mid-structure.

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