Smoothness, Fairness and the need for better multi-sided patches

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ABSTRACT. This paper surveys the key achievements and outstanding challenges of constructing smooth surfaces for geometric design. The focus here is on explicit methods in parametric form. In particular, recent insights into the curvature magnitude and distribution of surfaces generated by existing algorithms, based on generalized subdivision and on splines, are illustrated and corresponding research questions are formulated. These challenges motivate the search for alternative approaches to multi-sided patch constructions.

1. The need for multi-sided patches

If all smooth surfaces could be modeled by a checkerboard mesh such that every mesh node is surrounded by four quadrilaterals, we would simply parametrize them by tensor-product splines and the question, how to create everywhere smooth surfaces, would be simple to answer from standard spline theory [**dB87**]:use bidgree $k+1$ to obtain kth order smoothness. However, many surfaces have arbitrary local connectivity and global topological genus and such surfaces, meshes must, already by the Euler count, either include n-valent vertices where $n \neq 4$, or n-sided facets. Removal of the offending vertices or facets leads to holes in the mesh. Typically, we can assume that these holes are isolated, since there exist a number of refinement strategies of an input mesh that create only additional mesh nodes that are 4-valent (e.g. [**CC78**]). Since we can associate tensor-product splines with all the 4-valent nodes nodes, we are left with the task of *filling n-sided holes* in an otherwise smooth, regularly parametrized surface.

Key words and phrases. Differential geometry, surfaces, continuity, fairness, n-sided holes.
supported in part by NSF Grant #9457806-CCR..

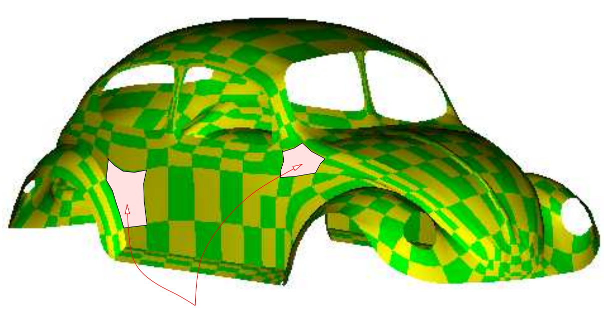

FIGURE 1. Multi-sided patches are needed to fill these holes.

There are numerous solutions to filling n-sided holes even with curvature continuity, starting with [**Hah89**], but, until recently, the polynomial or rational degrees of such approaches where dauntingly high (e.g. bi-degree (18,18) in [**Hah89**]) and thus a challenge for any downstream use, say the implicitization (ref other article in proceedings) or intersection of these surfaces.

Besides reducing the degree of everywhere smooth surfaces (see e.g. [**Pet02b**] for a survey of the appropriate notion of 'geometric smoothness'), fairness of the resulting surface is a major concern. It is typically easy to decide when a surface is not fair (see Figures 2,5). The positive statement is, however, much more difficult and contentious. Having an agreed upon, exact definition of when a surface is fair would be a big step towards constructive solutions. A rough consensus is that a surface is 'fair' if it has optimal monotonicity and distribution of curvature. (There are alternative definitions based on artistic perceptions or the availability of special mirrors [**Dis37**].)

Currently, the drive for higher surface quality is fueled from at least two sources. First, the new generation graphics cards allows us to notice even minor flaws in high-quality surfaces. Second, due to faster turn-around times, industrial geometric design faces increasing time pressure when 'cleaning up' product-defining outer surfaces such as 'class A' outer panel surfaces in automobile design. Surprisingly, tweaking control structures, by teams of second-tier designers, is still common practice. Of special concern are regions where several primary surfaces meet, say where the roof support meets the hood, the side and the fender region of a car (Figure 1). Much time is spent in fitting and trimming a single NURBS surface patch (with continuity up to a tolerance!) to n-sided holes representing the common join of n primary surfaces. Similar challenges arise the design of headlight reflectors where the overall available space and position is defined by the layout of the gear train and the aesthetic design, but a given functionality, in terms of the light cone swept out, has to be achieved.

2. A study of the state of the art

While the practice of pulling single or groups of spline control points appears antiquated, a reliable automatic alternative does not seem to be available. To check this, the author and K. Karciauskas conducted an intensive study in 2001/2002 (partly presented at the St Malo conference on Curves and Surfaces in June 2002), reimplementing all known polynomial and rational, curvature continuous, quadrilateral-patch based schemes, as well as several variants of Catmull-Clark subdivision, and constructions such as [**Kar99**], similar to toric patches [**Kra02, KK00**].

We especially scrutinized curvature distribution near points where $n \neq 4$ patches come together. As pointed out above, this is the key, as well as the hardest modeling scenario since here, the bivariate nature of the modelling problem comes to the fore. We developed an obstacle course of boundary conditions that would naturally occur in the design setting – such as joints in a Y configuration and the U shape (Figure 6) – and flagged schemes that

(a) clearly betrayed patch boundaries under curvature or highlight scrutiny (Figure 2, *top*, *left*),
(b) displayed oscillations in curvature not warranted by the boundary data (Figure 2, *top*, *right*, *bottom*),
(c) exaggerated rather than attenuated curvature variation (Figure 6) or even converged to the wrong curvature (e.g. a subdivision started on a convex mesh exhibiting hyperbolic behavior in the limit).

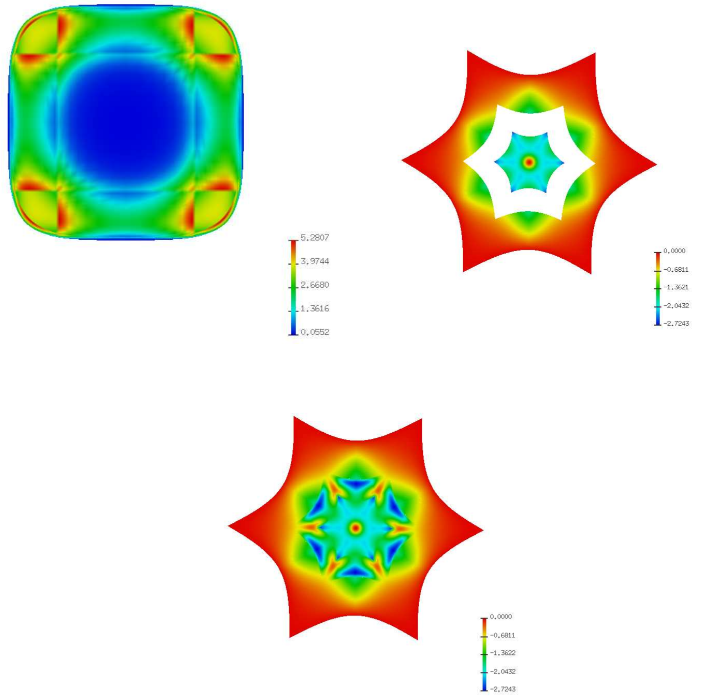

FIGURE 2. *Best viewed in color!* Commonly encountered deficiencies of piecewise polynomial C^2 surface constructions, emphasized by Gauss curvature texture. (*upper left*) Sharp transitions in curvature reveal the patch boundaries. (*upper right*) With the transition layer removed (white hexagonal gap), the Gauss curvature of the center group of patches and the outer annulus is monotone. All shape problems (oscillations) have been pushed into the transition layer as is visible by re-inserting the layer of patches (*bottom*).

Even though we applied a wide variety of linearized functionals advertised in the literature to pin down extra degrees of freedom, we typically found a host of deficiencies *near* such n-sided configurations. Here, it is important to understand that blending to a predefined, smooth shape or composing with a simple smooth shape, such as a quadratic [**Pra97, Rei98**], only *exports the fairness challenge to the transition layer* (see Figure 2).

The study left, however, hope for certain bidegree six or higher degree polynomial schemes and a new class of rational schemes related to toric patches (see e.g. Figure 5).

3. Current approaches to filling multi-sided holes in surfaces

A central challenge of research into the design of smooth surfaces is to understand the construction of fair, curvature continuous surfaces well enough to *no longer treat n-sided holes as an exception*. This is the crucial step beyond the theory of tensor-product, or, more generally, box-spline constructions.

In this section, we review two of the most popular approaches and their shortcomings, and open questions.

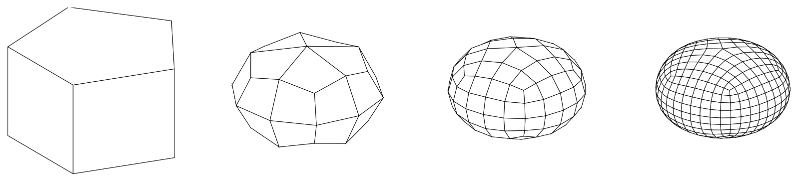

FIGURE 3. Three steps of Catmull-Clark subdivision applied to an extruded pentagon.

The focus is on parametric representations with built-in smoothness that can be computed non-iteratively; that is, to determine control points, singular value decomposition is acceptable but not solving a global partial differential equation; energy functionals should be built into the construction, not be the result of an optimization process as in anisotropic diffusion. Building smoothness and fairness into the representation leads to higher efficiency through smaller constraint systems, higher accuracy and stability than if the constraints were added to, say, a fine triangle mesh. It also allows accurate measurement of all shape quantities to be optimized. Finally, such an explicit representation can serve as a good starting point for more expensive shape-improving computations.

3.1. Generalized Subdivision. For the last few years, progress in modeling has been associated with generalized subdivision surfaces [**CC78, DS78, PR97, Loo87, Kob00, WW01**] as a natural averaging paradigm. The conceptual simplicity of subdivision surfaces has taken the entertainment industry by storm [**DKT98**] adding an additional representation to all standard graphics design packages.

Yet, at present, no one seems to be ready to base a full car body design or the crafting of special purpose lenses on this paradigm. Indeed, detailed analysis has revealed severe problems for high-quality design with surfaces generated by standard subdivision algorithms both of shape in the large and shape near so-called extraordinary points. An extraordinary point is the limit point when filling n-sided holes in surfaces by an infite sequence of piecewise polynomial rings (Figure 4). It appears that while subdivision initially does a fine job in smoothing out transitions between primary surfaces, in the limit the problems are concentrated in the neighborhood of the extraordinary node. Also, subdivision based on box splines necessarily inherits both the advantages and the flaws of polynomial surfaces and shape problems familiar from spline schemes reappear. We observed that the distribution of curvature deteriorate progressively as 'the wrong' eigenfrequencies take over (c.f. Figure 6). More smoothing of the regular regions appears to be actually detrimental to the shape near these extraordinary points [**ZS01**]. Algorithms modified to have bounded curvature near extraordinary n-valent points exhibit oscillations in the curvature (Figure 5, *lower left*).

3.1.1. *Artifacts in the large.* The need to improve the shape of subdivision surfaces was already recognized in [**HKD93**]. In " Tutorials on multiresolution in geometric modeling" [**IQF02**], Malcolm Sabin pointed out ". . . high singularities and the combination of singularities can cause patterns of artifact around the singularity. . . . in the Catmull-Clark case take a 12-sided prism, about as high as it is long; in the Loop case take a pyramid with a 12-sided base. In both cases the initial polyhedron is convex but after a couple of refinements a shape is produced which is not even convex. No univariate scheme is that bad. There is going to be a significant amount of work required to work this out, and it has not really been started yet."

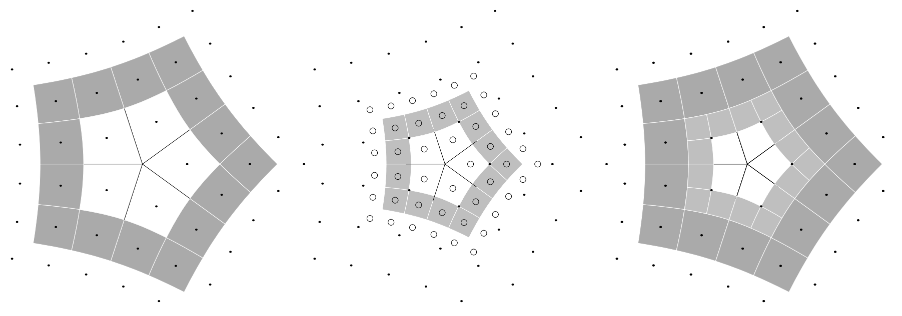

FIGURE 4. Subdivision filling an n-sided hole by an infinite sequence of spline rings (*dots* and *circles* represent spline control points).

3.1.2. *Artifacts in the small.* Asymptotic analysis leads to strong necessary conditions for nontrivial second order differentiability at extraordinary points: the higher-order eigenfunctions, associated with the eigenvectors of the subdivision matrix for generating rings of patches, have to be quadratic polynomials of the subdominant eigenfunctions (see e.g. [**WW01, Rei99**]). For piecewise polynomial basis functions, this quadratic dependence implies a lower bound on the degree of the polynomials. Therefore most researchers conclude that there is little hope that C^2 schemes will be as natural and elegant as the standard schemes. In many applications this flaw is excused by arguing that standard subdivision surfaces appear to be fair enough and not far from C^2. Even divergence of curvature, as observed, for example, for Catmull-Clark subdivision surfaces, is accepted (Figure 6).

As in the spline case, one can sidestep problems at the extraordinary point by either blending with a prescribed shape (say a quadratic) or composing with this shape. However, just as in the spline case, this just moves the shape problems to the transition layer between a central core and the surrounding regular (box-)spline surface (c.f. Figure 2). Typical problems of subdivision schemes are as follows.

– Unequal contraction of the subdivision mesh (parametrization) depending on the valence rather than the geometry of a mesh node;
– Lack of monotonicity (c.f. Figure 6) and shape preservation.

Some key questions to be answered in the future are as follows.

S1 Do there exist variants of Catmull-Clark subdivision that produce fair shapes in the vicinity of extraordinary nodes?
S2 Do bounded curvature schemes (see e.g. [**Loo01, Sab91**]) inherently lead to oscillations in the curvature ?
S3 Are there curvature-bounded schemes that have small masks and arise naturally without explicit manipulation of the spectrum?
S4 Are subdivision schemes good preprocessors for other schemes to fill n-sided holes? Or do they necessarily worsen the scenario as indicated in Figure 6?
S5 Can shape deficiencies in the large, e.g. lack of monotonicity, be reduced using anisotropic subdivision?

3.2. Spline techniques. Many subdivision schemes are useful because they are derived from and inherit properties of (box-)splines [**dBHR93, PB02, WW01**], i.e. piecewise polynomial functions. Here, however, we are interested in surfaces that consist of a finite number of polynomial pieces that are connected with geometric continuity [**Pet02b**].

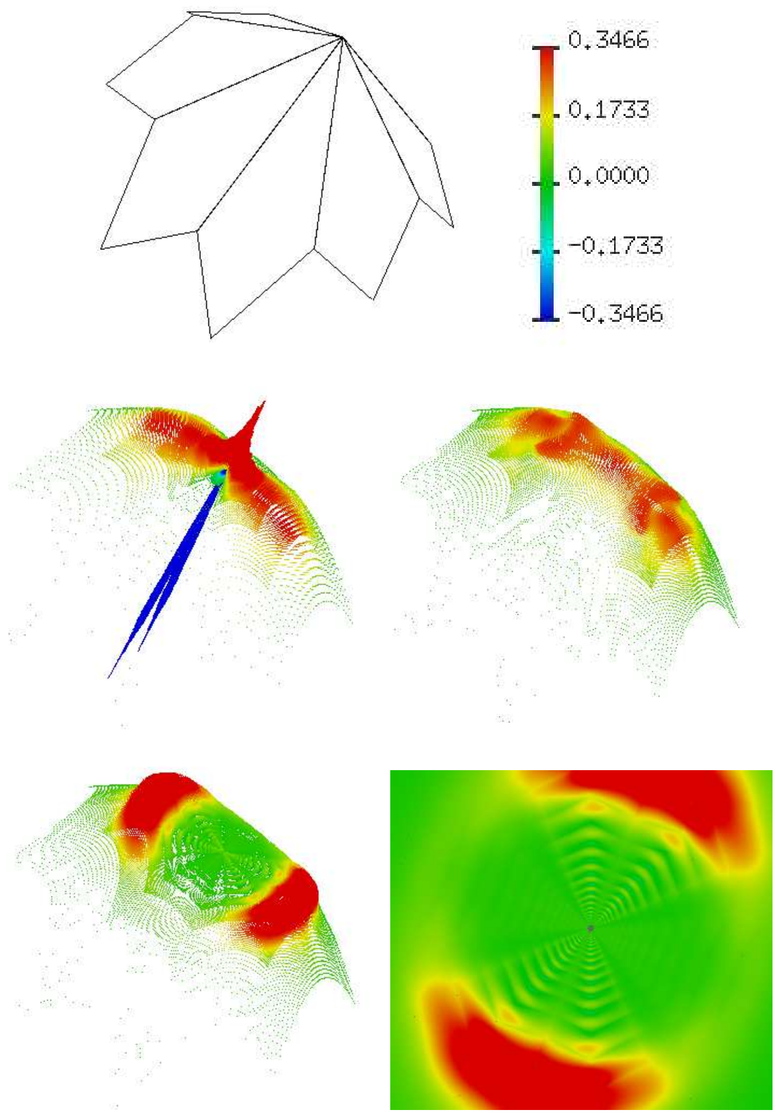

FIGURE 5. *Best viewed in color!* (*top*) A parabola-like control net and Gauss curvature scale used to texture the figures below. (*middle left*) Needle plot representing Gauss curvature of the Catmull-Clark subdivision surface corresponding to the input net. The curvature diverges. (*middle right*) Gauss needle plot of a new construction based on toric blends; (*bottom:*) bounded version of Catmull-Clark [**Sab91**] with enlargement of central region to illustrate the curvature oscillation.

High quality surfaces require continuity of curvatures, while efficient computation, say of intersections, favors polynomial representations of low degree. In practice, curved shapes with a tensor-product layout are therefore typically designed using splines of degree $(3,3)$ or $(5,5)$. For arbitrary patch layout, however, known surface constructions of degree

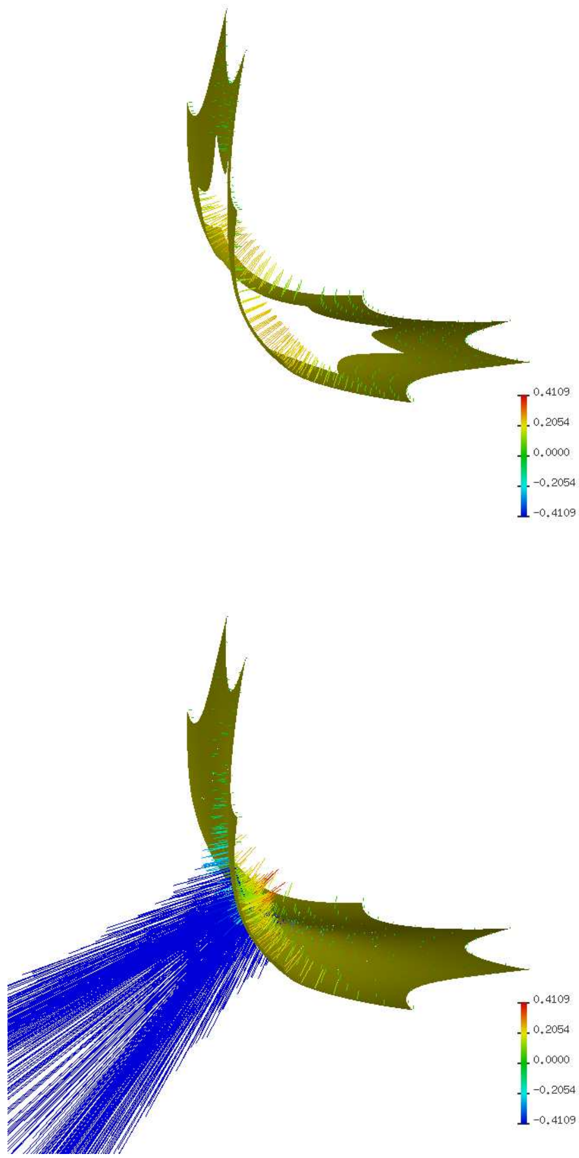

FIGURE 6. (*top*) Side view of a ring of bicubic spline patches following an extruded U valley. The length of the needles in the normal direction is proportional to the Gauss curvature. Note that the Gauss curvature is positive for the ring. (*bottom*) The hole filled by several iterations of Catmull-Clark subdivision. Note that, as indicated by the change of direction and increased length of the needles, the Gauss curvature becomes negative and increases (without bound).

$(3,3)$ are only tangent continuous (see e.g. [**GZ94, Pet94, Rei95, Pet00**]); and it seems unlikely that curvature continuity can be achieved using just bicubic patches except if one is willing to accept shape deficiencies such as flat regions. (As we saw in the previous section, even if we use an infinite sequence of bicubic patches near the extraordinary points, as in the Catmull-Clark generalized subdivision scheme [**CC78**], we obtain at best bounded curvature [**Sab91**].)

With the exception of [**GZ99, Pet02a**], curvature continuous polynomial constructions using tensor-product patches are of degree $(9,9)$ or higher [**Hah89, GH89, Ye97**] or they use an approach developed by Prautzsch and Reif [**Pra97, Rei98**]: a disk-shaped region in $\mathbb{R}^2$ defined by a C^k tensor-product spline of degree $(k+1, k+1)$ is mapped onto a polynomial surface of total degree d in $\mathbb{R}^3$ to cover the neighborhood of an extraordinary point by a surface of degree $(dk+d, dk+d)$. If $k = 2$ and $d = 2$, the degree is as low as (6,6); but, at *higher-order saddle points*, $d = 2$ leads to undesirable flat patches rather than zero Gaussian curvature at the central point and negative Gaussian curvature in the immediate neighborhood. Quadratic boundary curves in [**GZ99**] lead to similar shape deficiencies. Increasing d to 3, as suggested in [**Rei99**] to avoid flat patches, leads again to patches of degree $(9,9)$. [**PU00, BR97, Pet96**] explain constructions using three-sided patches, the last as low as total degree 8. C^2 surfaces can be constructed from the ideas in [**GH95, LD89, NG00**] using rational patches of degree $(9,9)$ and higher. [**Du88, Her96**] use second order 'Gregory patches' of degree $(5,5)$ that are singular in the corners and have more than twice the number of coefficients of the construction in [**Pet02a**]: new techniques in [**Pet02a**] yield curvature continuous surfaces consisting of splines of degree $(3,3)$ plus some strips of splines of degree $(3,5)$.

Experiments strengthen the conjecture that a good parametrization is the key to applying linearized functionals effectively. Here [**Pet02a**] does not fare well, since it distorts the natural alignment of parameter lines as n becomes large. The bicubic reparametrizations in [**Pra97, Rei98, GZ99, NG00**], on the hand, yield parametrizations similar to Catmull-Clark. The most promising approach for polynomial splines, in terms of degree and lack of algorithm-induced shape deficiencies is therefore to hybridize the techniques in [**Pet02a**] with those in [**GZ99**].
Some key questions to be answered in the future are as follows.

- P1 Do piecewise polynomial schemes inherently betray their patch structure through curvature artifacts that align with parameter lines?
- P2 What is the polynomial precision of spline schemes on a topological annulus surrounding an n-sided hole? Are spline schemes able to reproduce polynomials on a n-sided annulus in $\mathbb{R}^3$?

The last, approximation theoretic question may be central to the shape deficiencies we observe both for spline and for subdivision schemes. It should be noted that approximation theory is well-developed for functions; it is less clear how notions like approximation-order and polynomial reproduction are to be applied to parametric surfaces (although arc length parametrization has been successfully used to transfer notions from univariate functions to curves, e.g. [**dBHS87**]). One possible approach is to ask whether a scheme has enough 'flexibility' in that it can represent hyperbolic and elliptic shapes when the Gauss curvature of the data surrounding the n-sided hole are elliptic or hyperbolic, i.e. have positive or negative curvature.

4. Field-based approach and toric patches

The above exposition showed that the the fundamental and hard challenge of *creating artifact-free surfaces that withstand scrutiny in the detail* is far from being met. Keeping in mind that many applications require parametric, finite, affine invariant methods, two other directions of enquiry still merit further exploration. First, can a surrounding field guide the parametric surface towards a variational minimum? Since the resolution is finite, and since the required representation has an associated parametrization, this minimum may not be efficiently approached or the approximation may have very different analytic properties. Second, can we leverage toric patch techniques, reducing the rational degree for matching surrounding C^2 data and tie the patches into mainstream representations? This is explored in other articles of this collection.

We close by observing that research into fair, curvature-continuous surface design promises to continue to be one of the hardest challenges of applied differential geometry, geometric design and graphics – and toric techniques and patch constructions have the potential to effectively address that core challenge.

References

[BR97] H. Bohl and U. Reif. Degenerate Bézier patches with continuous curvature. *Comput. Aided Geom. Design*, 14(8):749–761, 1997.

[CC78] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-Aided Design*, 10:350–355, September 1978.

[dB87] Carl de Boor. B-Form Basics. In Gerald Farin, editor, *Geometric Modeling : Algorithms and New Trends*, pages 131–148, Philadelphia, 1987. SIAM.

[dBHR93] C. de Boor, K. Höllig, and S. Riemenschneider. *Box splines*, volume 98 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[dBHS87] Carl de Boor, Klaus Hollig, and Malcolm Sabin. High accuracy geometric hermite interpolation. *Computer Aided Geometric Design*, 4(4):269–278, December 1987.

[Dis37] W. Disney. Snow white and the seven dwarfs. *Disney Publications*, 1937. "Mirror,mirror on the wall,. . . ", see also Grimm's Fairy Tales 1823.

[DKT98] Tony DeRose, Michael Kass, and Tien Truong. Subdivision surfaces in character animation. In Michael Cohen, editor, *SIGGRAPH 98 Conference Proceedings*, Annual Conference Series, pages 85–94. ACM SIGGRAPH, Addison Wesley, July 1998. ISBN 0-89791-999-8.

[DS78] D. Doo and M. Sabin. Behaviour of recursive division surfaces near extraordinary points. *Computer-Aided Design*, 10:356–360, September 1978.

[Du88] Wen-Hui Du. *Etude sur la représentation de surfaces complexes: application à la réconstruction de surfaces échantillonnées*. PhD thesis, Ecole Nationale Supeŕieure des Télécommunications, Paris, France, 1988.

[GH89] John A. Gregory and Jörg M. Hahn. A C^2 polygonal surface patch. *Comput. Aided Geom. Design*, 6(1):69–75, 1989.

[GH95] C. M. Grimm and J. F. Hughes. Modeling surfaces of arbitrary topology using manifolds. *Computer Graphics*, 29(Annual Conference Series):359–368, 1995.

[GZ94] John A. Gregory and Jianwei Zhou. Filling polygonal holes with bicubic patches. *Computer Aided Geometric Design*, 11(4):391–410, 1994.

[GZ99] John A. Gregory and Jianwei Zhou. Irregular C^2 surface construction using bi-polynomial rectangular patches. *Comput. Aided Geom. Design*, 16(5):423–435, 1999.

[Hah89] Jörg Hahn. Filling polygonal holes with rectangular patches. In *Theory and practice of geometric modeling (Blaubeuren, 1988)*, pages 81–91. Springer, Berlin, 1989.

[Her96] T. Hermann. G^2 interpolation of free-form curve networks by biquintic Gregory patches. *Comput. Aided Geom. Design*, 13:873–893, 1996.

[HKD93] Mark Halstead, Michael Kass, and Tony DeRose. Efficient, fair interpolation using Catmull-Clark surfaces. In James T. Kajiya, editor, *Computer Graphics (SIGGRAPH '93 Proceedings)*, volume 27, pages 35–44, August 1993.

[IQF02] A. Iske, E. Quak, and M. Floater, editors. *Tutorials on Multiresolution in Geometric Modeling*. Springer Verlag, 2002.

[Kar99] K. Karciauskas. Rational multisided sabin and hosaka-kimura like surface patches. Technical Report 99-31, Department of Mathematics and Informatics, Vilnius University, 1999. (pdf: 696KB).

[KK00] R. Krasauskas K. Karciauskas. Comparison of different multisided patches using algebraic geometry. In P. Sablonniere P.J. Laurent and L.L. Schumaker, editors, *Curve and Surface Design: Saint-Malo, 1999*, pages 163–172. Vanderbilt University Press, Nashville, 2000.

[Kob00] Leif Kobbelt. $\sqrt{3}$ subdivision. In Sheila Hoffmeyer, editor, *Proceedings of the Computer Graphics Conference 2000 (SIGGRAPH-00)*, pages 103–112, New York, July 23–28 2000. ACMPress.

[Kra02] R. Krasauskas. Toric surface patches. Technical Report 2000-21, Faculty of Mathematics and Informatics, Vilnius University, 2002.

[LD89] Charles Loop and Tony D. DeRose. A multisided generalization of bezier surfaces. *ACM Transactions on Graphics*, 8(3):204–234, July 1989.

[Loo87] C.T. Loop. Smooth Subdivision Surfaces Based on Triangles. Master's thesis, Department of Mathematics, University of Utah, August 1987.

[Loo01] C. Loop. Triangle mesh subdivision with bounded curvature and the convex hull property. Technical report, MSR Tech Report MSR-TR-2001-24, 2001.

[NG00] J. Cotrina Navau and N. Pla Garcia. Modeling surfaces from meshes of arbitrary topology. *Comput. Aided Geom. Design*, 17(7):643–671, 2000.

[PB02] H. Prautzsch and W. Boehm. *Handbook of Computer Aided Geometric Design, eds. G. Farin and J. Hoschek and M.-S. Kim*, chapter 10 Box splines, pages 255–283. North-Holland, New York, 2002.

[Pet94] J. Peters. Surfaces of arbitrary topology constructed from biquadratics and bicubics. In Nickolas S. Sapidis, editor, *Designing fair curves and surfaces: shape quality in geometric modeling and computer-aided design*, pages 277–293. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1994.

[Pet96] J. Peters. Curvature continuous spline surfaces over irregular meshes. *Computer-Aided Geometric Design*, 13(2):101–131, Feb 1996.

[Pet00] J. Peters. Patching Catmull-Clark meshes. *Siggraph 2000*, pages 255–258, 2000.

[Pet02a] J. Peters. C^2 free-form surfaces of degree (3,5). *Computer Aided Geometric Design*, 19(2):113–126, 2002.

[Pet02b] J. Peters. Geometric continuity. In *Handbook of Computer Aided Geometric Design*. Elsevier, 2002.

[PR97] J. Peters and U. Reif. The simplest subdivision scheme for smoothing polyhedra. *ACM Transactions on Graphics*, 16(4):420–431, October 1997.

[Pra97] Hartmut Prautzsch. Freeform splines. *Comput. Aided Geom. Design*, 14(3):201–206, 1997.

[PU00] H. Prautzsch and G. Umlauf. Triangular G^2splines. In L.L. Schumaker P.J. Laurent, A. LeMéhauté, editor, *Curve and Surface Design*, pages 335–342. Vanderbilt University Press, 2000.

[Rei95] Ulrich Reif. Biquadratic G-spline surfaces. *Computer Aided Geometric Design*, 12(2):193–205, 1995.

[Rei98] U. Reif. TURBS—topologically unrestricted rational B-splines. *Constr. Approx.*, 14(1):57–77, 1998.

[Rei99] Ulrich Reif. *Analyse und Konstruktion von Subdivisionsalgorithmen für Freiformflächen beliebiger Topologie*. Shaker Verlag, Aachen, 1999.

[Sab91] Malcolm Sabin. Cubic recursive division with bounded curvature. In L.L. Schumaker P.J. Laurent, A. LeMéhauté, editor, *Curves and Surfaces*, pages 411–414. Academic Press, 1991.

[WW01] Joe Warren and Henrik Weimer. *Subdivision Methods for Geometric Design: A Constructive Approach*. Morgan Kaufmann Publishers, 2001.

[Ye97] Xiuzi Ye. Curvature continuous interpolation of curve meshes. *Computer Aided Geometric Design*, 14(2):169–190, 1997.

[ZS01] Denis Zorin and Peter Schr¨oder. A unified framework for primal/dual quadrilateral subdivision schemes. *Comput. Aided Geom. Design*, 18(5):429–454, 2001. Subdivision algorithms (Schloss Dagstuhl, 2000).

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