On the Optimality of Piecewise Linear Max-norm Enclosures based on Slefes

Jörg Peters and Xiaobin Wu

Abstract. Subdividable linear efficient function enclosures (Slefes) provide, at low cost, a piecewise linear pair of upper and lower bounds f^+, f^-, that sandwich a function f on a given interval: $f^+ \geq f \geq f^-$. In practice, these bounds are observed to be very tight. This paper addresses the question just how close to optimal, in the max-norm, the slefe construction actually is. Specifically, we compare the width $f^+ - f^-$ of the slefe to the narrowest possible piecewise linear enclosure of f when f is a univariate cubic polynomial.

§1. Introduction

Due to curved geometry, objects in b-spline, Bézier or generalized subdivision representation pose numerical and implementation challenges when measuring distance between objects, re-approximating for format conversion, meshing with tolerance, or detecting the silhouette. Naive linearization, say triangulation by sampling, reapproximates without known error and not safely from one side. Subdividable linear efficient function enclosures (slefes) [7,8,9], by contrast, are a low-cost technique yielding two one-sided, piecewise linear bounds that sandwich nonlinear functions. The width of a slefe, i.e., the distance between upper and lower approximation, is easily measured, because it is taken on at a breakpoint, and refinement yields predictably tighter enclosures.

Slefe-based bounds are observed to be very tight. Yet, being linear, the slefe construction cannot be expected to provide the best two-sided max-norm approximation. Therefore, it is of interest to see how close to optimal the slefe construction actually is by deriving and comparing it with the narrowest possible enclosure. Since slefes generate the minimal width enclosures for quadratics, we focus on univariate cubic polynomial pieces.

Curve and Surface Design: Saint-Malo 2002

Tom Lyche, Marie-Laurence Mazure, and Larry L. Schumaker (eds.), pp. 335–344.

ISBN 0-9728482-0-7.

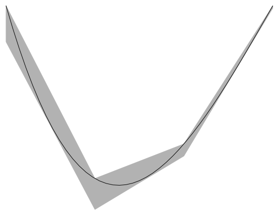

Fig. 1. A cubic Bézier segment inside its grey-shaded slefe.

1.1. Prior Work

The theory of slefes has its roots in bounds on the distance of piecewise polynomials to their Bézier or b-spline control net [10,12]. Compared to these constructions, enclosures yield dramatically tighter bounds for the underlying functions since they do not enclose the control polygon. Approximation theory has long recognized the problems of one-sided approximation and two-sided approximation [1]. Algorithmically, though, according to the seminal monograph [11] p. 181, the convergence of the proposed Remez-type algorithms is already in one variable 'generally very slow'. The only termination guarantee is that a subsequence must exist that converges. By contrast, the slefes provide a solution with an explicit error very fast and with a guarantee of error reduction under refinement.

The object oriented bounding boxes for subdivision curves or surfaces in [6] are based on a min–max criterion and require the evaluation of several points and normals on the curve or surface. Thus, the dependence on the coefficients is not linear. Linearity of the slefe construction is highly desirable since it allows us to solve hard inverse problems, such as fitting spline curves into prescribed channels. Farin [2] shows that for rational Bézier–curves, the convex hull property can be tightened to the convex hull of the first and the last control point and so-called *weight* points. Hu et al. [3,4,5,14] promote the use of *interval spline representation* (see Farouki and Sederberg [13]) for tolerancing, error maintenance and data fitting. The key ingredient of this use of interval arithmetic are axis-aligned bounding boxes based on the positivity and partition of unity property of the b-splines. Enclosures complement this work by offering tighter two-sided bounds.

§2. Subdividable Linear Efficient Function Enclosures

The slefe of a function f with respect to a domain U is a piecewise linear pair, f^+, f^-, of upper and lower bounds that sandwich the function on U: $f^+ \geq f \geq f^-$. Here, we focus on piecewise linear f^+ and f^- and measure the width, $f^+ - f^-$, in the *recursively applied* L^∞ *norm*: the width is as small as possible where it is maximal – and, having fixed the breakpoint

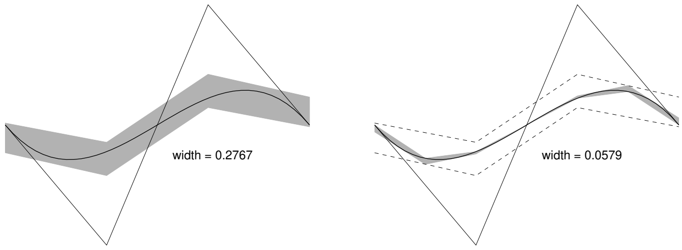

Fig. 2. (*left*) A cubic Bézier segment with coefficients $0, -1, 1, 0$. The control polygon exaggerates the curve far more than the *grey* 3-piece slefe. (*right*) After one subdivision at the midpoint, the width of the slefe (*grey*) is roughly 1/4th of the width of the unsubdivided slefe (*dashed*) .

values where the maximal width is taken on (zeroth and first breakpoint in Fig. 1), the width at the remaining breakpoints is recursively minimized subject to matching the already fixed break point values.

Slefes are based on the two general lemmas [7,9] and the once-and-for-all tabulation of best recursive L^∞ enclosures for a small set of functions, $\mathbf{a} := (\mathbf{a}_i)_{i=1,\dots,s}$ below.

Lemma 1. *Given two finite-dimensional vector spaces of functions,* $\mathcal{B} \neq \mathcal{H}$, $s := \dim \mathcal{B} - \dim(\mathcal{B} \cap \mathcal{H})$, $(\mathbf{b}_i)_{i=1,\dots,\dim \mathcal{B}}$ *a basis of* $\mathcal{B}$, $(\mathbf{a}_i)_{i=1,\dots,s}$ *functions in* $\mathcal{B}$, *the embedding identity* $E : \mathcal{B} \to \mathcal{B} + \mathcal{H}$, *and linear maps*

$$L : \mathcal{B} \to \mathcal{H}, \quad \Delta : \mathcal{B} \to \mathbb{R}^s,$$

such that (i) $(\Delta_j \mathbf{a}_i)_{i,j}$ *is the identity in* $\mathbb{R}^{s \times s}$ *and (ii)* $\ker \Delta = \ker(E - L)$, *then for any* $f = \mathbf{b} \cdot \mathtt{f} := \sum \mathbf{b}_i \mathtt{f}_i \in \mathcal{B}$,

$$(\mathbf{b} - L\mathbf{b}) \cdot \mathtt{f} = (\mathbf{a} - L\mathbf{a}) \cdot (\Delta f).$$

Remarks: For practical computation, $(\mathbf{a} - L\mathbf{a}) \cdot (\Delta f)$ has to have only finitely many terms, e.g. $s < \infty$. Items (i) and (ii) make $E - \mathbf{a}\Delta$ a projector into a space invariant under L. In (ii), $\ker \Delta \subset \ker(E - L)$ is needed since for any $f \in \ker \Delta \setminus \ker(E - L)$, $(\mathbf{a} - L\mathbf{a}) \cdot (\Delta f)$ is zero, but not $(\mathbf{b} - L\mathbf{b}) \cdot \mathtt{f}$. Since the width of the enclosure changes under addition of any element in $\ker(E - L) \setminus \ker \Delta$, we also want $\ker(E - L) \subset \ker \Delta$.

Lemma 2. *If, with the definitions of Lemma 1, additionally the maps* $\lfloor \cdot \rfloor, \lceil \cdot \rceil : \mathcal{B}^s \to \mathcal{H}^s$ *satisfy* $\lfloor \mathbf{a} - L\mathbf{a} \rfloor \leq \mathbf{a} - L\mathbf{a} \leq \lceil \mathbf{a} - L\mathbf{a} \rceil$ *pointwise and componentwise, and* $(\Delta f)_+(i) := \max\{0, \Delta f(i)\}$, $(\Delta f)_-(i) := \min\{0, \Delta f(i)\}$ *then*

$$f^- := Lf + \lfloor \mathbf{a} - L\mathbf{a} \rfloor \cdot (\Delta f)_+ + \lceil \mathbf{a} - L\mathbf{a} \rceil \cdot (\Delta f)_-,$$

$$f^+ := Lf + \lfloor \mathbf{a} - L\mathbf{a} \rfloor \cdot (\Delta f)_- + \lceil \mathbf{a} - L\mathbf{a} \rceil \cdot (\Delta f)_+$$

sandwich f, *i.e.,* $f^- \leq f \leq f^+$.

Example. Let $\mathcal{B}$ be the space of univariate polynomials of degree d, say in Bézier form, Δf the $d-1$ second differences of its Bézier coefficients, $U = [0..1]$ and $\mathcal{H}$ the space of piecewise linear functions with break points at i/m, $i \in \{0, \ldots, m\}$. To obtain slefes, we determine and tabulate the breakpoint values of $\lfloor \mathbf{a}_i - L\mathbf{a}_i \rfloor_m \in \mathcal{H}$ and $\lceil \mathbf{a}_i - L\mathbf{a}_i \rceil_m \in \mathcal{H}$ that minimize the width

$$w_{\text{slefe}}(f;U) := \max_U f^+ - f^- = \max_U \sum_{i=1}^{d-1} (\lceil \mathbf{a}_i - L\mathbf{a}_i \rceil_m - \lfloor \mathbf{a}_i - L\mathbf{a}_i \rfloor_m) \, |\Delta_i f|.$$

The width is invariant under addition of constant and linear terms to f and one (DeCasteljau) subdivision step at the midpoint, $t = 1/2$ cuts the width to roughly *a quarter* (see Figure 2).

The general *slefe construction* is as follows. (Note that (0),(1),(2),(3) are precomputed, off-line and (4) is cheap, making the computation of slefes efficient.)

(0) Choose U, the domain of interest, and the space $\mathcal{H}$ of enclosure functions.

(1) Choose a difference operator $\Delta : \mathcal{B} \mapsto \mathbb{R}^s$, with $\ker \Delta = \mathcal{B} \cap \mathcal{H}$.

(2) Compute $\mathbf{a} : \mathbb{R}^s \mapsto \mathcal{B}$ so that $\Delta\mathbf{a}$ is the identity on $\mathbb{R}^s$ and each $\mathbf{a}_i$ matches the same $\dim(\mathcal{B} \cap \mathcal{H})$ additional independent constraints.

(3) Compute $\lfloor \mathbf{a} - L\mathbf{a} \rfloor$ and $\lceil \mathbf{a} - L\mathbf{a} \rceil \in \mathcal{H}$.

(4) Compute $(\Delta f)_+$ and $(\Delta f)_-$ and assemble f^- and f^+ (Lemma 2).

§3. Optimal Bounds for Cubic Functions

In this section, we determine, for a class of functions, the optimal enclosure width and compare it with w_{slefe}. The simplest nontrivial case is when the function f is a univariate quadratic polynomial; however, in this case, the slefe construction is optimal, because the vector of functions $\mathbf{a} - L\mathbf{a}$ is a singleton and slefes are based on the optimal enclosures of $\lfloor \mathbf{a} - L\mathbf{a} \rfloor, \lceil \mathbf{a} - L\mathbf{a} \rceil$. Since explicit determination of the least-width piecewise linear enclosure is a challenge, we consider polynomials f of degree $d = 3$ in Bézier representation on the interval $U = [0..1]$:

$$f(u) := \textstyle\sum_{i=0}^d f_i \mathbf{b}_i^d(u), \quad \mathbf{b}_i^d(u) := \frac{d!}{(d-i)!i!}(1-u)^{d-i}u^i.$$

We approximate from the space of hat functions with $m = 3$ segments and breakpoints at j/m, $j = 0, \ldots, m$. Generalization of the results to $m > 3$ pieces is not difficult; generalization to degree $d > 3$ has not yet been attempted. Without loss of generality, we assume

$$\Delta_1 f \geq |\Delta_2 f|, \qquad \text{where} \quad \Delta f := \begin{bmatrix} \Delta_1 f \\ \Delta_2 f \end{bmatrix} := \begin{bmatrix} f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \end{bmatrix}.$$

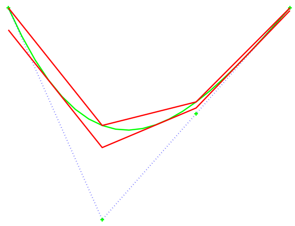

Fig. 3. $a := \mathbf{a}_1^3 := -(2\mathbf{b}_1^3 + \mathbf{b}_2^3)/3$ with control polygon and slefe.

3.1. Computing w_{slefe}

With $Lf(u) := f_0(1-u) + f_d u$, we have

$$\mathbf{a}_1^d := -(2\mathbf{b}_1^d + \mathbf{b}_2^d)/3, \quad \mathbf{a}_2^d := -(\mathbf{b}_1^d + 2\mathbf{b}_2^d)/3,$$

and $\mathbf{a}_i^d - L\mathbf{a}_i^d = \mathbf{a}_i^d$. One checks Lemma 1:

$$f - Lf = \mathbf{a}_1^d \Delta_1 f + \mathbf{a}_2^d \Delta_2 f.$$

The optimal enclosures for $(\mathbf{a}_j^3)_{j=1,2}$ have been tabulated; but here, we derive them as explicit symbolic expressions. By symmetry, it is sufficient to compute bounds for $a := \mathbf{a}_1^3$. Due to the convexity of a (see Fig. 3), the piecewise linear interpolant at j/m is an upper bound. We express $\lceil a \rceil$ as the vector of its breakpoint values (e.g. the value of $\lceil a \rceil$ at $1/3$ is $-10/27$):

$$27\lceil a \rceil \simeq [0, -10, -8, 0].$$

The lower bound is computed by recursive minimization. The first segment is the dominant segment in the sense that its tightest bound has the largest width among the three segments (Fig. 3, see also Lemma 5). Therefore, we calculate the values of $\lfloor a \rfloor$ at 0 and 1/3 by shifting down the first segment of the upper bound until it is tangent to a. The other two break point values are computed by calculating the tangent line to a, keeping one end fixed. This procedure yields the four break point values of the lower bound

$$27\lfloor a \rfloor \simeq [30, 20, 25 + \frac{\beta_1 - 9}{2}\beta_2, \beta_3] - \frac{38\beta_1}{9},$$

where

$$\beta_1 := \sqrt{57}, \quad \beta_2 := \sqrt{-10 + 2\beta_1},$$

$$\beta_3 := \frac{261}{8} + \frac{\beta_1 - 9}{4}\beta_2 + \frac{3\beta_2 - \beta_1}{8}\sqrt{11 - 12\beta_2 - 2\beta_1 + 2\beta_1\beta_2}.$$

An approximation of the values is $\lfloor a \rfloor \approx [-.0695, -.4399, -.3154, -.0087]$ The width of a is

$$w_{\text{slefe}}(a) = w_{\text{opt}}(a) = -\frac{10}{9} + \frac{38\beta_1}{243} \approx 0.0695.$$

Let $w_i := (\lceil a \rceil - \lfloor a \rfloor)(i/m)$, $i = 0,1,2,3$. Then, due to the symmetry $\mathbf{a}_1^3(t) = \mathbf{a}_2^3(1-t)$,

$$w_{\text{slefe}}(f;U) = \max_{i\in\{0,1,2,3\}} \{|\Delta_1 f| w_i + |\Delta_2 f| w_{m-i}\}.$$

Since $w_0 = w_1 > w_2 > w_3$, the term with $i = 1$ is the maximal term, and

$$w_{\text{slefe}}(f;U) = |\Delta_1 f| w_1 + |\Delta_2 f| w_2 \approx 0.0695|\Delta_1 f| + 0.0191|\Delta_2 f|.$$

If we set $|\Delta_1 f| := 1 + \epsilon$, $\epsilon \in [0..\infty]$, and $|\Delta_2 f| := 1$, then

$$w_{\text{slefe}}(f;U) = -\frac{1}{243}(270\epsilon + 567 + \frac{9\beta_2}{2}(\beta_1 - 9) - 38\beta_1(\epsilon + 2)).$$

3.2. Computing w_{opt}

We next determine w_{opt}, the width of the *narrowest* possible piecewise linear enclosure for f with break points at $i/3$ (that is, $w_{\text{opt}}(f,[0..1/3])$ based on enclosing by one linear segment above and one below and $w_{\text{opt}}(f,[0..1])$ on three). The next three lemmas show that (i) it is sufficient to compare the width of functions with first and last coefficient equal zero; (ii) an increase of the second derivative of f then increases w_{opt}; (iii) if $|\Delta_1 f| > |\Delta_2 f|$ then the first segment determines w_{opt}.

Lemma 3. *Let ℓ be a linear function, $U = [0..1]$ and $f'' > 0$ on U. Then $w_{\text{opt}}(f;U) = w_{\text{opt}}(f+\ell;U)$, and the t^* at which the width is taken on is the same for f and $f + \ell$.*

Proof: Due to convexity,

$$\begin{aligned} w_{\text{opt}}(f+\ell;U) &= \max_t (1-t)(f(0)+\ell(0)) + t(f(1)+\ell(1)) - (f(t)+\ell(t)) \\ &= \max_t (1-t)f(0) + tf(1) - f(t) = w_{\text{opt}}(f;U). \quad \square \end{aligned}$$

Lemma 4. *Let $U = [0..1]$, $f(0) = f(1) = g(0) = g(1) = 0$, $f'' > 0$ on U, $g'' = f'' + c$, where $c > 0$ is a constant. Then $w_{\text{opt}}(g;U) > w_{\text{opt}}(f;U)$.*

Proof: $g - f = -\frac{c}{6}(\mathbf{b}_1^3 + \mathbf{b}_2^3) < 0$ on $(0..1)$. $\square$

Lemma 5. *If $\Delta_1 f > |\Delta_2 f|$ then $w_{\text{opt}}(f,[0..1/3]) \geq w_{\text{opt}}(f,[0..1])$.*

Proof: Let $w := w_{\text{opt}}(f,[0..1/3])$ be the minimal width of the first segment, and

$$\sigma_i^+ := \begin{cases} 1, & \text{if } f''(\frac{i}{3}) > 0 \\ 0, & \text{else} \end{cases}, \qquad \sigma_i^- := \begin{cases} 1, & \text{if } f''(\frac{i}{3}) < 0 \\ 0, & \text{else.} \end{cases}$$

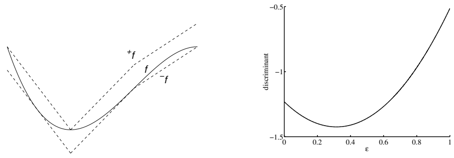

Fig. 4. (*left*) The enclosure (not a slefe!) used to prove dominance of the left-most segment. (*right*) The discriminant of the intersection problem is negative .

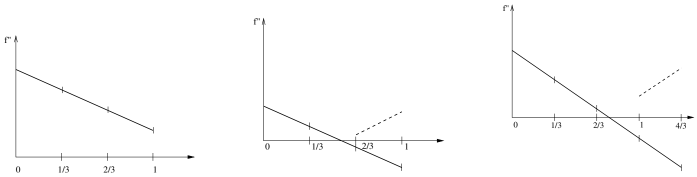

Fig. 5. (*left*) Linear $f'' > 0$, (*middle*) inflection in the middle segment, (*right*) inflection in the right segment (middle segment with respect to $U' := [1/3..4/3]$) .

We show that the piecewise linear function ${}^{+}f$ with breakpoint values (Fig. 4, *left*) $f(\frac{i}{3})+\sigma_i^- w, i = 0, \ldots, 3$, yields and upper bound and ${}^{-}f$ with values $f(\frac{i}{3}) - \sigma_i^+ w, i = 0, \ldots, 3$, yields a lower bound.

For segments without inflection point, this follows from Lemma 4. Since f'' is linear, all segments have the same slope (see Fig. 5, *left*). The leftmost segments is larger by a positive constant, because $\Delta_1 f > |\Delta_2 f|$. In purely concave segments, Lemma 4 applies to the flipped derivative (see Fig. 5, *middle, dashed* line segment). Now if $\Delta_2 f < 0$, let

$$\Delta_1 f := 1 + \epsilon, \quad \epsilon \in [0, 1], \quad \text{and} \quad \Delta_2 f := -1.$$

Since $\Delta_1 f > |\Delta_2 f|$, there are no inflections in the interval $U_1 := [0..1/2]$. If $\epsilon \in [0..1]$ then the inflection is in the interval $U_2 := [1/2..2/3]$ and if $\epsilon > 1$ then the inflection is in the interval $U_3 := [2/3..1]$.

If the inflection point belongs to the middle segment, we show that the line segment connecting $(1/3, f(1/3) - w)$ to $(2/3, f(2/3))$ only intersects $(t, f(t))$ at $2/3$; similarly, $(1/3, f(1/3))$ to $(2/3, f(2/3) + w)$ only intersects $(t, f(t))$ at $1/3$. In the first case, the quadratic resulting from canceling the factor $(t - 2/3)$ of $f(t) - (f(1/3) - w)(1 - t) - f(2/3)t$ has the discriminant

$$\frac{1}{(\epsilon + 2)87723}(-2888A^{3/2} + (17001 + 24795\epsilon)A - 175689 - 355023\epsilon,$$

where $A := 57\epsilon^2 + 93\epsilon + 39$. For $\epsilon \in [0..1]$, the discriminant plotted in Fig. 4, *right* attains a maximum of

$$28 - 56/9\sqrt{21} \approx -.51380432.$$

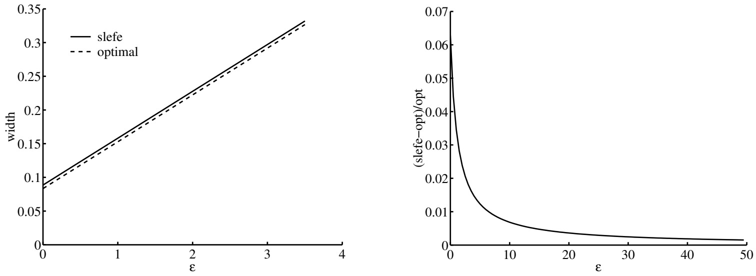

Fig. 6. Convex cubic f. (*left*) Value of $w_{\text{slefe}}(f;U)$ *vs* $w^{\cup}_{\text{opt}}(f;U)$. (*right*) The ratio $\frac{w_{\text{slefe}}-w^{\cup}_{\text{opt}}}{w^{\cup}_{\text{opt}}}(f;U)$.

Similarly, the maximum in the second case is $25/9 - 52/81\sqrt{39} \approx -1.231$. This lack of additional roots proves that ${}^{+}f$ and ${}^{-}f$ are an upper and a lower bound respectively.

If $\epsilon > 1$, we can shift U to $[1/3..4/3]$ and the segment $U_3 = [2/3..1]$ can be treated as a middle segment (c.f. Fig. 5,*right*). $\square$

3.3. Comparison of w_{opt} and w_{slefe}

By symmetry, we assume that $\Delta_1 f \geq |\Delta_2 f|$. If $\Delta_2 f = 0$ (and $f(0) = f(1) = 0$) then f is a multiple of a_1^3 and $w_{\text{opt}} = w_{\text{slefe}}$. Also, both w_{opt} and w_{slefe} scale linearly with $\Delta_2 f$. We therefore normalize in the following so that $|\Delta_2 f| = 1$.

We first consider $w_{\text{opt}} = w^{\cup}_{\text{opt}}$, the case of *no inflection.* Without loss of generality, $\Delta_1 f := 1 + \epsilon, \epsilon \in [0, \infty]$, $\Delta_2 f := 1$ and with $A := \sqrt{57\epsilon^2 + 135\epsilon + 81}$, we compute

$$w^{\cup}_{\text{opt}}(f;U) := -\frac{1}{243}\frac{(9+9\epsilon - A)(-3A(1+\epsilon) + 11\epsilon^2 + 36\epsilon + 27)}{\epsilon^2}.$$

Figure 6, *left* plots w_{slefe} against $w^{\cup}_{\text{opt}}$. The gap between w_{slefe} and $w^{\cup}_{\text{opt}}$ increases with ϵ but is finite at infinity:

$$(w_{\text{slefe}}(f) - w^{\cup}_{\text{opt}}(f))(\epsilon = \infty) = \frac{16}{9} - \frac{\beta_1}{54}\beta_2 + \frac{\beta_2}{6} - \frac{59\beta_1}{243} \approx .0053353794.$$

The relative difference has a maximum of ca. 6% when $\epsilon = 0$ (c.f. Fig. 6,*right*), i.e. when f is of degree 2.

If f has an inflection point, we may assume that $\Delta_1 f := 1 + \epsilon$, $\epsilon \in [0, \infty]$, and $\Delta_2 f := -1$ and compute $w_{\text{opt}} = w^{\sim}_{\text{opt}}$,

$$w^{\sim}_{\text{opt}}(f;U) = -\frac{1}{243}\frac{(9\epsilon + 9 - A)(-3A(1+\epsilon) + 11\epsilon^2 + 8\epsilon - 1)}{(\epsilon+2)^2}$$

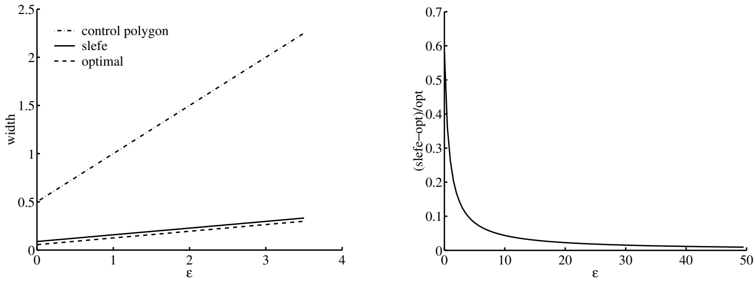

Fig. 7. Cubic f with inflection: (*left*) Value of $w_{\text{slefe}}(f;U)$, $w_{\text{opt}}^{\sim}(f;U)$ and the width based on the convex hull of the control polygon. (*right*) The ratio $\frac{w_{\text{slefe}}-w_{\text{opt}}^{\sim}}{w_{\text{opt}}^{\sim}}(f;U)$.

where $A := \sqrt{57\epsilon^2 + 93\epsilon + 39}$. Now

$$(w_{\text{slefe}}(f) - w_{\text{opt}}^{\sim}(f))(\epsilon = \infty) = \frac{9\beta_2 + 30\beta_1 - 228 - \beta_1\beta_2}{54} \approx .032775216.$$

The worst ratio $\frac{w_{\text{slefe}}-w_{\text{opt}}^{\sim}}{w_{\text{opt}}^{\sim}}(f;U)$ occurs when f is of the type depicted in Figure 2: if $\Delta_1 f = -\Delta_2 f = 1$ then $w_{\text{opt}}^{\sim}(f) = .05593616039$ and $w_{\text{slefe}}(f) = .08857673214$. Although the ratio is almost 3:5, the slefe is considerably tighter than the convex hull of the control polygon (c.f. Fig. 7, *left*).

Acknowledgments. This research was made possible in part by the grant NSF 9457806-CCR

References

1. Buck, R. C., Applications of duality in approximation theory, in *Approximation of Functions* (Proc. Sympos. General Motors Res. Lab., 1964), Elsevier Publ. Co., Amsterdam, 1965, 27–42.
2. Farin, G., Tighter convex hulls for rational Bézier curves, Comput. Aided Geom. Design **10**, 1993, 123–125.
3. Hu, C.-Y., T. Maekawa, E. C. Sherbrooke, and N. M. Patrikalakis, Robust interval algorithm for curve intersections, Computer-Aided Design **28**, 1996, 495–506.
4. Hu, C.-Y., N. M. Patrikalakis, and X. Ye, Robust interval solid modelling part I: representations, Computer-Aided Design **28**, 1996, 807–817.

5. Hu. C.-Y., Ni. M. Patrikalakis, and X. Ye, Robust interval solid modelling part II: boundary evaluation, Computer-Aided Design **28**, 1996, 819–830.
6. Kobbelt, L. P., K. Daubert, and H.-P. Seidel, Ray tracing of subdivision surfaces, in *Rendering Techniques* '98 (Proceedings of the Eurographics Workshop), Springer-Verlag, New York, 1998, 69–80.
7. Lutterkort, D., *Envelopes for Nonlinear Geometry*, PhD thesis, Purdue University, May 2000.
8. Lutterkort, D. and J. Peters, Linear envelopes for uniform B–spline curves, Curves and Surfaces, St Malo, 2000, 239–246.
9. Lutterkort, D. and J. Peters, Optimized refinable enclosures of multivariate polynomial pieces, Comput. Aided Geom. Design **19**, 2002, 851–863.
10. Nairn, D., J. Peters, and D. Lutterkort, Sharp, quantitative bounds on the distance between a polynomial piece and its Bézier control polygon, Comput. Aided Geom. Design **16**, 1999, 613–633.
11. Pinkus, A. M., *On L^1-approximation*, Cambridge University Press, Cambridge, 1989.
12. Reif, U., Best bounds on the approximation of polynomials and splines by their control structure, Comput. Aided Geom. Design **17**, 2000, 579–589.
13. Sederberg, T. W. and R. T. Farouki, Approximation by interval Bézier curves, IEEE Computer Graphics and Applications **12**, 1992, 87–95.
14. Tuohy, S. T., T. Maekawa, G. Shen, and N. M. Patrikalakis, Approximation of measured data with interval B-splines, Computer-Aided Design **29**, 1997, 791–799.

J. Peters and X. Wu
Dept. of CISE
University of Florida
Gainesville, FL 32611
jorg@cise.ufl.edu
http://www.cise.ufl.edu/~jorg