Computing curvature bounds for bounded curvature subdivision*

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Abstract

For a stationary, affine invariant, symmetric, linear and local subdivision scheme that is C^2 apart from the extraordinary point the curvature is bounded if the square of the subdominant eigenvalue equals the subsubdominant eigenvalue. This paper gives a technique for quickly establishing an interval to which the curvature is confined at the extraordinary point. The approach factors the work into precomputed intervals that depend only on the scheme and a mesh-specific component. In many cases the intervals are tight enough to be used as a test of shape-faithfulness of the given subdivision scheme; i.e. if the limit surface in the neighborhood of the extraordinary point of the subdivision scheme is consistent with the geometry implied by the input mesh.

1 Introduction

It is well known that the classic subdivision schemes [Catmull & Clark '78, Loop '87] have either undefined or zero curvature at their extraordinary points. Since numerical procedures are sensitive to such singularities, [Sabin '91] and later [Holt '96, Loop '00] proposed subdivision schemes that guarantee a bounded though not necessarily convergent curvature. These improved subdivision schemes are obtained by a judicious perturbation of the subdivision masks of the classic subdivision schemes so that the square of the subdominant eigenvalue equals the subsubdominant eigenvalue [Doo & Sabin '78, Ball & Storry '88, Reif '93]. Ball and Storry estimated curvature along symmetrically placed curves on surfaces generated by variants of Catmull/Clark subdivision [Ball & Storry '90].

For practical use it is important not just to have some curvature bound but to know that the bound is commensurable with the input data and how to quickly compute it in an actual setting. While the value of the bound must depend on the input mesh, we would like a subdivision scheme to be closely related to a measure of curvature of the input mesh. This paper shows how to compute good bounds quickly based on a factorization of the work into a precomputed table of intervals that depend only on the specific scheme, and a mesh-specific component. With the bounds we can compare schemes for their reproduction of mesh shapes, a tool for qualitatively comparing subdivision schemes. For example it is good if we can prove that a mesh representing a convex polyhedron results in a surface with positive Gaussian curvature.

As an example we derive interval tables for Gaussian and mean curvature for a curvature bounded variant of the Loop algorithm and a curvature bounded variant of the Catmull-Clark algorithm.

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2 Curvature of subdivision surfaces

We consider stationary, affinely invariant, symmetric, linear and local subdivision schemes such as [Catmull & Clark '78, Loop '87, Kobbelt '99, Velho '00] that generate limit surfaces that are C^2 everywhere except at a small number of extraordinary points where the normal is only continuous. In the vicinity of an extraordinary mesh node with n neighbors such schemes can be described by a square, stochastic subdivision matrix A that maps a coarse control mesh $\mathcal{C}_{m-1}$ to a finer mesh $\mathcal{C}_m = A\mathcal{C}_{m-1}, m \geq 1$. We make following assumptions on the eigenstructure of A:

- The eigenvalues of A are:

$$1 = \lambda_0 > \underbrace{\lambda_1 = \lambda_2}_{=:\,\lambda} > \underbrace{\lambda_3 = \lambda_4 = \lambda_5}_{=:\,\mu} > \cdots \geq 0$$

  with right eigenvectors $\mathbf{r}_i : A\mathbf{r}_i = \lambda_i \mathbf{r}_i$ and left eigenvectors $\mathbf{l}_i : \mathbf{l}_i A = \lambda_i \mathbf{l}_i$ that satisfy $\mathbf{l}_i \mathbf{r}_j = \delta_{ij}$.

- In terms of the Fourier decomposition of A the eigenvalues
  - λ_0, λ_5 are from frequency block 0,
  - λ_1 is from frequency block 1, λ_2 is from frequency block $n-1$
  - λ_3 is from frequency block 2, λ_4 is from frequency block $n-2$.

The assumptions pick out the generic, geometrically desirable case, where exactly three subsub-eigenfunctions influence the curvature behavior at the extraordinary point. Many other, less symmetric cases are mathematically possible as detailed in [Reif '98, Zorin '98]. (These lead to a less complex geometry and simpler estimates than the ones below, say if $\lambda_3 > \lambda_4$, and hence $\mathbf{e}_4$ and $\mathbf{e}_5$ do not contribute to the curvature, we get $K = \mathcal{P}_{33}\mathcal{D}_{33}$.)

In the m-th iteration a surface ring $\mathbf{x}_m$ is generated that can be parameterized in terms of the basis functions $B(u,v)$ for regular meshes:

$$\mathbf{x}_m : \begin{cases} \{1,\ldots,n\} \times \Omega \to \mathbb{R}^3, \\ (i,u,v) \mapsto B(u,v)\mathcal{C}_m. \end{cases}$$

The domain segments Ω are shown in Figure 1. Therefore each surface ring can be expressed as

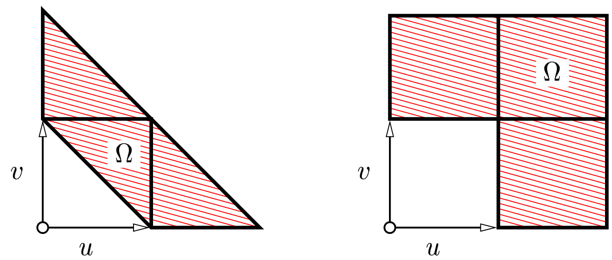

Figure 1: The domain segments Ω.

$\mathbf{x}_m = \sum \lambda_i^m \mathbf{e}^i(u,v)\mathbf{p}_i$ with eigenfunctions

$$\mathbf{e}^i : \begin{cases} \{1,\ldots,n\} \times \Omega \to \mathbb{R}, \\ (i,u,v) \mapsto B(u,v)\mathbf{r}_i \end{cases}$$

and coefficients $\mathbf{p}_i = \mathbf{l}_i \mathcal{C}_0$ of the eigen-expansion of the input mesh. According to [Peters & Umlauf '00] the possibly diverging sequences of Gaussian and mean curvature when approaching the extraordinary point have the formal limit expression

$$K = \sum_{i,j=3,4,5} \mathcal{P}_{ij}\mathcal{D}_{ij}, \qquad H = \sum_{\substack{i=3,4,5\\ k,l=1,2,\ k\geq l}} \mathcal{P}_{ikl}\mathcal{D}_{ikl}$$

where

$$\begin{aligned}
\mathcal{P}_{ij} &:= \frac{\det(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_i)\det(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_j)}{\|\mathbf{p}_1\times\mathbf{p}_2\|^4}, & \mathcal{P}_{ikl} &:= \frac{\det(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_i)(\mathbf{p}_k\mathbf{p}_l^t)}{\|\mathbf{p}_1\times\mathbf{p}_2\|^3},\\
\mathcal{D}_{ij} &:= \frac{D^i_{uu}D^j_{vv} - D^i_{uv}D^j_{uv}}{\Delta_{12}^4}, & \mathcal{D}_{ikl} &:= \frac{\varepsilon_{kl}(D^i_{uu}(\mathbf{e}^k_v\mathbf{e}^l_v) - D^i_{uv}(\mathbf{e}^k_u\mathbf{e}^l_v + \mathbf{e}^k_v\mathbf{e}^l_u) + D^i_{vv}(\mathbf{e}^k_u\mathbf{e}^l_u))}{\Delta_{12}^3},\\
\Delta_{ij} &:= \mathbf{e}^i_u\mathbf{e}^j_v - \mathbf{e}^j_u\mathbf{e}^i_v, & \varepsilon_{kl} &:= 1/2 \text{ for } k=l, 1 \text{ otherwise},\\
D^i_{st} &:= \Delta_{12}\mathbf{e}^i_{st} - \Delta_{1i}\mathbf{e}^2_{st} + \Delta_{2i}\mathbf{e}^1_{st}, & & s,t\in\{u,v\},\\
\mathbf{e}^i_s &:= \partial\mathbf{e}^i/\partial s, \qquad \mathbf{e}^i_{st} := \partial^2\mathbf{e}^i/\partial s\partial t, & & s,t\in\{u,v\}.
\end{aligned}$$

We refer to $\mathcal{D}_{ij}$ and $\mathcal{D}_{ikl}$ as D-factors and their multipliers $\mathcal{P}_{ij}$ and $\mathcal{P}_{ikl}$ as P-factors. Scaling of $\mathbf{r}_i$ does not change K and H but just the ratio between the D- and P-factors. Since the D-factors depend only on the subdivision scheme they can be precomputed and tabulated for one fixed scale. Also note that $\mathcal{P}_{ij} = \mathcal{P}_{ji}$ so that three pairs of D-factors contributing to K can be combined. The D-factors determine the limits of the curvature sequences when approaching the extraordinary point from different directions.

Our goal is to bound the D-factors. The multiplications and differentiations may be expensive. However, because of rotational symmetry of the eigenfunctions it suffices to compute the D-factors only for the first domain segment $\{1\}\times\Omega$. Since the expressions in the denominators of the D-factors are identical for all domain segment $\{i\}\times\Omega, i=1,\dots,n$, it is convenient to separately bound the denominators $\Delta_{12}^3, \Delta_{12}^4$ and the numerators of $\mathcal{D}_{ij}+\mathcal{D}_{ji}$, $i\neq j$, $\mathcal{D}_{ii}$ and $\mathcal{D}_{ikl}$,

$$\begin{aligned}
G_{ij} &:= D^i_{uu}D^j_{vv} - 2D^i_{uv}D^j_{uv} + D^i_{vv}D^j_{uu} && \text{for } i,j\in\{3,4,5\},\ j>i,\\
G_{ii} &:= D^i_{uu}D^i_{vv} - (D^i_{uv})^2 && \text{for } i=3,4,5,\\
G_{ikl} &:= D^i_{uu}(\mathbf{e}^k_v\mathbf{e}^l_v) - D^i_{uv}(\mathbf{e}^k_u\mathbf{e}^l_v + \mathbf{e}^k_v\mathbf{e}^l_u) + D^i_{vv}(\mathbf{e}^k_u\mathbf{e}^l_u) && \text{for } i=3,4,5, k,l=1,2, k>l.
\end{aligned}$$

These expressions for the domain segments $\{i\}\times\Omega, i=2,\dots,n$ can be obtained as linear combinations of the respective expressions for $\{1\}\times\Omega$. For example $G_{33}|_{\{i\}\times\Omega}$ is given by,

$$G_{33}|_{\{i\}\times\Omega} = \alpha^2 G_{33}|_{\{1\}\times\Omega} - \alpha\beta G_{34}|_{\{1\}\times\Omega} + \beta^2 G_{44}|_{\{1\}\times\Omega}, \tag{1}$$

where $\alpha = \cos(2i\pi/n), \beta = \sin(2i\pi/n)$. We denote the minimal interval that includes G_{ij} on $\{1,\dots,n\}\times\Omega$ by $[\![G_{ij}]\!]$. Since the intervals of the denominators for all domain segments are equal, the interval of the D-factors are obtained by the quotient rule of interval arithmetic.

The $\mathbf{p}_i$ and hence the P-factors are computed using the precomputed left eigenvectors $\mathbf{l}_i$. This yields the intervals for K and H

$$K \in \sum_{i,j=3,4,5} \mathcal{P}_{ij}\cdot[\![\mathcal{D}_{ij}]\!], \qquad H \in \sum_{\substack{i=3,4,5\\ k,l=1,2,\ k\geq l}} \mathcal{P}_{ikl}\cdot[\![\mathcal{D}_{ikl}]\!].$$

Rather than computing the bounds exactly, it is often sufficient to compute an approximation. For example, if the underlying basis is polynomial then the extremal Bézier coefficients can serve as bounds. Tighter bounds can be obtained by refining the representation of G_{ij}, G_{ikl} or by the general construction of [Lutterkort '99] for refinable functions. Also, while expressing the surface rings $\mathbf{x}_m$ in terms of scaled eigenfunctions $\sigma_i \mathbf{e}^i$ does not change the product of the D- and P-values, but only their relative contribution, and therefore does not alter the *exact* bounds on K and H, *approximate* bounds can be affected by scaling before rotation. In particular, we found that we can improve, say the min-max bounds of the Bézier representation, by scaling the complex right eigenvectors $\mathbf{r}_1 + \mathbf{r}_2\sqrt{-1}$ and $\mathbf{r}_3 + \mathbf{r}_4\sqrt{-1}$ with a complex number and choosing the intersection of the intervals. The computational costs of this improvement are small because the expressions of the new numerators are linear combinations of the old expressions $G_{ij}|_{\{1\}\times\Omega}$ respectively $G_{ikl}|_{\{1\}\times\Omega}$ analogous to (1).

3 Two curvature bounded subdivision schemes

We tested the technique on a curvature bounded variant of the Loop algorithm and a curvature bounded variant of the Catmull-Clark algorithm.

The Loop variant as proposed in [Prautzsch & Umlauf '98b, Umlauf '99] is defined by the masks in Figure 2 with the weights

$$\begin{aligned}
\alpha &= \frac{c_1^2}{16} + \frac{3c_1}{16} + \frac{33}{64}, \quad c_i := \cos(2i\pi/n), \\
\alpha_i &= f_i + \frac{2}{n}\sum_{j=1}^{k} \delta_j c_{i(j+1)}, \; i = 0, \ldots, \lfloor n/2 \rfloor, \\
\delta_1 &= \lambda^2 - \mu, \qquad \delta_i \; < \; \lambda^2 - \lambda_i, \qquad i \geq 2, \\
f_i &= \begin{cases} 3/8 & i = 0 \\ 1/8 \quad \text{if} & i = 1 \\ 0 & i \geq 2 \end{cases} .
\end{aligned}$$

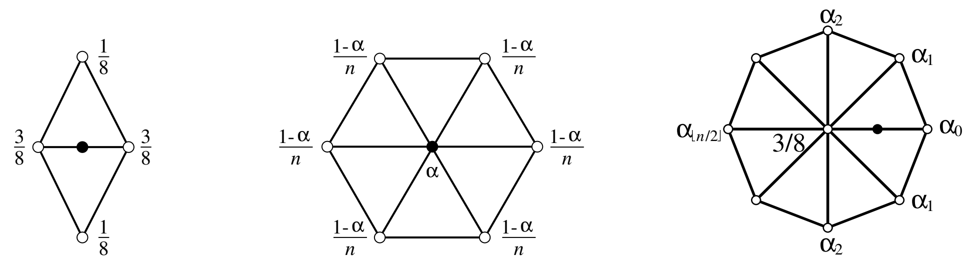

Figure 2: The masks of the curvature bounded variant of the Loop algorithm.

We normalize the right eigenvectors such that their first non-zero entry is 1. In case of a vertex of valence 7 we get the bounds for the D-factors in Table 1. This yields the following intervals for the Gaussian and mean curvature for a initial mesh with a vertex of valence 7 that lies on the parabolic cylinder $[u, v, u^2]$:

$$\begin{aligned}
K &\in [-53.628, 14.747] \\
H &\in [-0.491, 0.541].
\end{aligned}$$

The bounds for K can be reduced by the general fact $H^2 \geq K$.

D-Factor	lower bound	upper bound
$D_{3,3}$	-8.329	-2.417
$D_{3,4}$	-7.571	-2.516
$D_{3,5}$	-59.341	22.130
$D_{4,4}$	-0.475	1.594
$D_{4,5}$	-92.882	23.033
$D_{5,5}$	-370.057	101.762

D-Factor	lower bound	upper bound
$D_{3,1,1}$	-0.215	-0.069
$D_{3,1,2}$	-0.008	0.036
$D_{3,2,2}$	0.072	0.219
$D_{4,1,1}$	-0.022	0.056
$D_{4,1,2}$	-0.081	-0.018
$D_{4,2,2}$	0.050	0.027
$D_{5,1,1}$	-2.371	0.917
$D_{5,1,2}$	-1.214	1.044
$D_{5,2,2}$	-1.422	1.303

Table 1: Bounds for the D-factors of the curvature bounded variant of the Loop algorithm for $n = 7$ for K (left) and H (right).

Analogously we can also modify the Catmull-Clark scheme ([Catmull & Clark '78]) such that the subsubdominant eigenvalue equals the square of the subdominant eigenvalue, see [Prautzsch & Umlauf 98a] or [Umlauf '99, page 53 ff].

Again we normalize the right eigenvectors such that their first non-zero entry is 1. In case of a

D-Factor	lower bound	upper bound
$D_{3,3}$	-35.648	-1.060
$D_{3,4}$	-19.018	-1.270
$D_{3,5}$	-16.054	19.481
$D_{4,4}$	-0.107	24.204
$D_{4,5}$	-6.580	11.690
$D_{5,5}$	5.257	60.018

D-Factor	lower bound	upper bound
$D_{3,1,1}$	-8.266	-0.263
$D_{3,1,2}$	0.000	1.752
$D_{3,2,2}$	0.442	7.587
$D_{4,1,1}$	-2.787	3.021
$D_{4,1,2}$	-5.514	-0.846
$D_{4,2,2}$	-2.960	2.960
$D_{5,1,1}$	-11.855	-1.711
$D_{5,1,2}$	-12.116	10.833
$D_{5,2,2}$	-0.119	1.145

Table 2: Bounds for the D-factors of the curvature bounded variant of the Catmull-Clark algorithm for $n = 5$ for K (left) and H (right).

vertex of valence 5 we get the bounds for the D-factors in Table 2. Intersected with the intervals corresponding to the complex right eigenvectors with second non-zero entry set to 1 we obtain the following intervals for the Gaussian and mean curvature for an initial mesh with a vertex of valance 5 that lies on the parabolic cylinder $[u, v, u^2]$:

$$\begin{aligned} K &\in [5.329, 18.400] \\ H &\in [-5.997, 2.998]. \end{aligned}$$

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