Curvature of subdivision surfaces

— a differential geometric analysis and literature review —

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Motivation

Almost all subdivision algorithms in the current literature achieve \textit{tangent continuity but not curvature continuity}. \((C^1\text{ with infinite curvature!})\)
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Why is it difficult to achieve curvature continuity at an extraordinary point \((EOP)\)?
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Why is it difficult to achieve curvature continuity at an extraordinary point \((EOP)\)?

The quantities to measure are Gaussian and mean curvature in a neighborhood of an EOP!
Motivation

Almost all subdivision algorithms in the current literature achieve *tangent continuity but not curvature continuity*. ($C^1$ with infinite curvature!)

Why is it difficult to achieve curvature continuity at an extraordinary point (*EOP*)?

The quantities to measure are *Gaussian and mean curvature* in a neighborhood of an EOP!

**Sample result:**
At EOP the determinant of the *Jacobian of the subdominant* eigenfunctions of a curvature continuous subdivision algorithm must have *lower degree* than the determinant of the Jacobian of the surface.
Motivation: Review

Understand important *lower bound* results better:
Sabin 91, (≥ bi-4)
Reif 93,96, (≥ bi-6)
Prautzsch,Reif 99, (≥ bi-7(k + 1))
(Lower bounds on parametrization, not surface)

Understand *constructions* of curvature continuous piecewise polynomial subdivision algorithms
Prautzsch 97,
Prautzsch, Umlauf 98, Umlauf 99 (hybrid)
Reif 98.

Understand *stiffness* of such subdivision algorithms:
infinite collection of polynomial pieces
but generated by the same rule.
Talk Outline

- The (few) basics. (nomenclature)

- express curvatures of $m$th spline ring converging towards the EOP

\[ K_m = \left( \frac{\mu}{\lambda^2} \right)^2 m f^m_K(u, v), \quad H_m = \left( \frac{\mu}{\lambda^2} \right)^m f^m_H(u, v) \]

for scalar constants $\mu < \lambda$ and rational functions $f_K, f_H$.

$\mu/\lambda^2$: implies necessary constraints

Necessary and sufficient contraints: PDEs

- Lower bounds

- Prautzsch’s sufficient condition and construction.

- The key open problem! (well, sort of)

- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/
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K_m = (\mu / \lambda^2)^2 \text{ } f_K^m(u, v), \quad H_m = (\mu / \lambda^2)^m \text{ } f_H^m(u, v)
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The talk focusses generic subdivision (**GS**): generalization of \( C^2 \) box-spline subdivision generating regular \( C^1 \) surfaces; affine invariant, symmetric, linear, local, stationary. However applies to non-generic cases [Reif 98 (habil), Zorin 98 (thesis)] and non-polynomial cases.

Surface rings are box-splines (with basis \( B(u, v) \))

\[
\mathbf{x}_m: \{0, \ldots, n-1\} \times \Omega \rightarrow \mathbb{R}^3, \quad \mathbf{x}_m(u, v) = B(u, v) \mathbb{C}_m,
\]
Setting and definitions

\( A \) is square, stochastic **subdivision matrix**: \( C_m = A^m C_0 \), diagonalizable with eigenvalues

\[
1 = \lambda_0 > \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 = \lambda_5 > \cdots \geq 0,
\]

where \( \lambda_1 = \lambda_2 \) correspond to the 1st and \((n - 1)\)st block, \( \lambda_3 = \lambda_4 \) (for \( n > 3 \)) to the 2nd and \((n - 2)\)nd block and \( \lambda_5 \) to the 0th block of the Fourier decomposition of \( A \).

\( Av_i = \lambda_i v_i \) for all \( i \) yields eigendecomposition

\[
C_m = \sum_i \lambda_i^m v_i p_i, \quad p_i \in \mathbb{R}^3.
\]
Setting and definitions

Expanded in the *eigenfunction*

\[ \mathbf{e}^i : \{0, \ldots, n - 1\} \times \Omega \rightarrow \mathbb{R}, (u, v) \mapsto \mathcal{B}(u, v)\mathbf{v}_i \]

the surface ring \( \mathbf{x}_m \) is of the form

\[ \mathbf{x}_m(u, v) = \sum_i \lambda_i^m \mathcal{B}(u, v)\mathbf{v}_i \mathbf{p}_i = \sum_i \lambda_i^m \mathbf{e}^i(u, v)\mathbf{p}_i. \]
Gauss curvature $K$ and the mean curvature $H$ are

$$K(u, v) = \frac{e(u, v)g(u, v) - f(u, v)^2}{E(u, v)G(u, v) - F(u, v)^2},$$

$$H(u, v) = \frac{e(u, v)G(u, v) - 2f(u, v)F(u, v) + g(u, v)E(u, v)}{2(E(u, v)G(u, v) - F(u, v)^2)},$$

where

$$E = x_u x_u^t, \quad F = x_u x_v^t, \quad G = x_v x_v^t,$$

$$e = nx_{uu}^t, \quad f = nx_{uv}^t, \quad g = nx_{vv}^t,$$

and $n = (x_u \times x_v)/\|x_u \times x_v\|$ is the normal. Since $x$ is regular, $EG - F^2 = \|x_u \times x_v\|^2$ is nonzero and

$$K = \frac{\det(x_u, x_v, x_{uu}) \det(x_u, x_v, x_{vv}) - \det(x_u, x_v, x_{uv})^2}{\|x_u \times x_v\|^4},$$

$$H = \frac{\det(x_u, x_v, x_{uu}')(x_v x_v^t) - 2 \det(x_u, x_v, x_{uv}')(x_u x_v^t) + \det(x_u, x_v, x_{vv}')(x_u x_u^t)}{2\|x_u \times x_v\|^3}.$$
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Gauss curvature and mean curvature at EOP

Expand into eigenfunctions \( e^i \) as in [Reif 93]

\[
x_u = \lambda^m (e^1_u p_1 + e^2_u p_2) + \mu^m (e^3_u p_3 + e^4_u p_4 + e^5_u p_5) + o(\mu^m),
\]
\[
x_{uv} = \lambda^m (e^1_{uv} p_1 + e^2_{uv} p_2) + \mu^m (e^3_{uv} p_3 + e^4_{uv} p_4 + e^5_{uv} p_5) + o(\mu^m).
\]
\[
x_u \times x_v = \lambda^{2m} \Delta_{12}(p_1 \times p_2) + o(\lambda^{2m}),
\]
\[
\det(x_u, x_v, x_{uu}) = \lambda^{2m} \mu^m \sum_{i=3,4,5} \det(p_1, p_2, p_i) D^i_{uu} + o(\lambda^{2m} \mu^m).
\]

where

\[
\Delta_{ij} := e^i_u e^j_v - e^j_u e^i_v,
\]
\[
D^i_{st} := \Delta_{12} e^i_{st} - \Delta_{1t} e^2_{st} + \Delta_{2t} e^1_{st}, \quad s, t \in \{u, v\},
\]
\[
P_{ij} := \det(p_1, p_2, p_i) \det(p_1, p_2, p_j),
\]
\[ K_m = \left( \frac{\mu}{\lambda^2} \right)^{2m} \sum_{i,j=3,4,5} P_{ij} \left( D_{uu}^i D_{vv}^j - D_{uv}^i D_{uv}^j \right) + o(1) \]

\[ \Delta_{12}^4 \| p_1 \times p_2 \|^4 + o(1) \]

\( \Delta_{12} \) is the Jacobi determinant of the subeigenfunctions (‘characteristic map’). \( \| p_1 \times p_2 \| \) is positive for almost all initial control nets \( C_0 \). Hence denominator ok.

- If \( \mu > \lambda^2 \) then the Gauss curvature at the EOP is infinite. [Catmull-Clark 78, Loop 87, Qu 90]

- If \( \mu < \lambda^2 \) then the Gauss curvature at the EOP is zero. [Prautzsch & Umlauf ’98]

- If \( \mu = \lambda^2 \) then the Gauss curvature at the EOP is bounded by the second factor of \( K_m \) but is possibly non-unique [Sabin 91, Holt 96].

Note combination of tangent continuity and infinite curvature for \( \mu > \lambda^2 \).
If $\mu = \lambda^2$ then the limit for $m \to \infty$ yields at the EOP

$$K = \sum_{i,j=3,4,5} \frac{P_{ij}}{\|p_1 \times p_2\|^4} \frac{D_{uu}^i D_{vv}^j - D_{uv}^i D_{uv}^j}{\Delta_{12}^4}.$$ 

a rational function in $u$ and $v$ that **must be constant**!

$$P_{ij}(= P_{ji}) = \det(p_1, p_2, p_i) \det(p_1, p_2, p_j) \text{ arbitrary implies each summand has to be constant!}$$

Eigenfunctions $e^1, \ldots, e^5$ must satisfy the **six partial differential equations** (G-PDE):

$$D_{uu}^i D_{vv}^j - 2D_{uv}^i D_{uv}^j + D_{vv}^i D_{uu}^j = \Delta_{12}^4 \cdot \text{const}_{ij}, \quad \text{for } i, j \in \{3, 4, 5\}, \ j > i,$$

$$D_{uu}^i D_{vv}^i - (D_{uv}^i)^2 = \Delta_{12}^4 \cdot \text{const}_{ii}, \quad \text{for } i = 3, 4, 5.$$ 

**Summary** A GS has for almost all initial nets non-zero Gauss curvature at the EOP if and only if $\mu = \lambda^2$ and G-PDE holds. (9 additional partial differential equations for $H$)
General: GS is *curvature continuous* if $\mu = \lambda^2$ and the differential equations for $G$ and $H$ hold, because the *principal curvatures*

\[
\kappa_{1,2}^m = H_m \pm \sqrt{H_m^2 - K_m},
\]

converge like $O(\mu^m/\lambda^{2m})$ for $m \to \infty$.

Since $\int d\mathbf{x}_m = O(\lambda^{2m})$ and $\mu < \lambda$

\[
\sum_m \int_{x_m} |\kappa_{1,2}^m|^2 d\mathbf{x}_m = \sum_m O(\mu^{2m}/\lambda^{2m}) < \infty.
\]

which implies [Reif Schröder ’00] for $p = 2$: The principal curvatures of the limit surface of a GS are *square integrable*. 
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$\mu/\lambda^2$: implies necessary constraints

Necessary and sufficient contraints: \textbf{PDEs}

- Lower bounds

- Prautzsch’s sufficient condition and construction.

- The key open problem! (well, sort of)

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Lower bounds on the degree

**formal degree** vs **true degree**  \( \deg (= \text{number of non-constant derivatives}) \)

Recall Gauss PDE

\[
\begin{align*}
D_u^i D_v^j - 2D_u^i D_v^i D_{uv}^j + D_v^i D_{uv}^j &= \Delta_{12}^4 \cdot \text{const}_{ij}, \quad \text{for } i, j \in \{3, 4, 5\}, \ j > i, \\
D_u^i D_v^i - (D_u^i)^2 &= \Delta_{12}^4 \cdot \text{const}_{ii}, \quad \text{for } i = 3, 4, 5.
\end{align*}
\]

**Simple count** with \( d = \deg(x_0) \) total degree (resp. bi-degree) of regular parametrization. **Left side** of PDE

- total degree \( \leq 2(2(d - 1) + d - 2) = 6d - 8 \)
- bi-degree \( \leq 2(2d - 1 + d - 1) = 6d - 4 \),

whereas **right side** of PDE

- formal total degree of \( \Delta_{12}^4 \) is \( 4(2d - 2) \)
- formal bi-degree of \( \Delta_{12}^4 \) is \( 4(2d - 1) \).

**Degree mismatch**: (unless \( d = 0 \))

If the **true** degree equals the **formal** degree

then GS is curvature continuous if and only if \( \mu < \lambda^2 \),

i.e. EOP is a flat point.
A GS with $\mu = \lambda^2$ is curvature continuous only if the true degree of the Jacobian $\Delta_{12}$ is less than its formal degree! Options:

(i) The true degree of $e^1$ or $e^2$ is less than $d$.

(ii) The leading terms in the Jacobian $\Delta_{12}$ cancel.

If not (ii) and not flat then $d' := \deg(e^1) = \deg(e^2), d := \deg(x_0))$:

- Total degree $\deg(\text{left}_{i,j}) = 2(2d' + d - 4)$ and $\deg(\Delta^4_{12}) = 4(2d' - 2)$
- Bi-degree $\deg(\text{left}_{i,j}) = 2(2d' + d - 2)$ and $\deg(\Delta^4_{12}) = 4(2d' - 1)$.

Compare to find $2d' = d$:

If not (ii) then GS is curvature continuous and not flat only if the true (bi-)degree of the surface is at least twice the true (bi-)degree of the subdominant eigenfunctions $e^1$ and $e^2$. 
Comparison with earlier estimates

$2d' = d$ is consistent with *degree estimate of Reif 93, 96, Zorin 97*:

View surface as a function over the tangent plane parametrized by $e_1$ and $e_2$. Then non-flat implies non-tangential component at least quadratic in $e_1$ and $e_2$, i.e. $d \geq 2d'$.

[Prautzsch, Reif 99]
If the non-tangential component of the surface is at least of degree $r$ in $e_1$ and $e_2$ then the surface representation has to be at least of degree $rd'$. Since $e_1$ and $e_2$ have to have a minimal degree to form $C^k$ rings, e.g. $d' \geq k + 1$ in the tensor-product case, a lower bound is $r(k + 1)$.

(parametrization dependent reasoning about surfaces!)
Or – *(i)* the leading terms of $\Delta_{12}$ cancel

- **total degree:** $\deg(\text{left}_{i,j}) = 2 \max\{\deg(\Delta_{12}) + d - 2, 2(d - 1) + d - 2\} = 6d - 8$

- **bi-degree:** $\deg(\text{left}_{i,j}) = 2 \max\{\deg(\Delta_{12}) + d - 1, 2d - 1 + d - 1\} = 6d - 4$

Comparing with $\deg(\Delta_{12}^4) = 4 \deg(\Delta_{12})$.

If the true degree of $e^1$ and $e^2$ is not less than $d$ then GS is curvature continuous and not flat only if the total degree $\deg(\Delta_{12}) \leq 3d/2 - 2$, (*bi-degree*) $\deg(\Delta_{12}) \leq 3d/2 - 1$.

That is possible! E.g. if bi-$d = 4$ then $\deg(\Delta_{12}) = 5$ is needed as if $\deg(e^1) = \deg(e^2) = 3$
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Curvature continuous subdivision constructions

[Prautzsch & Umlauf ’98]:
induce flat spots to get low degree, small mask, curvature continuous subdivision algorithms.

[Sabin 91, Holt 96]:
adapt the leading eigenvalues to get non-zero bounded curvature.

Otherwise need degree-reduced Jacobian.

(Trivial) regular case of any $C^2$ box-spline: $e^1$ and $e^2$ are linear.

(Non-trivial) Projection of Prautzsch ’97, Reif ’98.
Sufficient conditions

\[ e^i = a_i(e^1)^2 + b_i e^1 e^2 + c_i(e^2)^2, \quad a_i, b_i, c_i \in R, \quad \text{for } i = 3, 4, 5. \]

Then (proof)

\[
\begin{align*}
e^i_u &= 2a^i e^1 e_u^1 + b^i (e_u^1 e^2 + e^1 e_u^2) + 2c^i e^2 e_u^2, \\
e^i_{uu} &= 2a^i ((e_u^1)^2 + e^1 e_{uu}^1) + b^i (e_{uu}^1 e^2 + 2e_u^1 e_u^2 + e^1 e_{uu}^2) + 2c^i ((e_u^2)^2 + e^2 e_{uu}^2) \\
\Delta_{1i} &= \Delta_{12} (b^i e^1 + 2c^i e^2), \\
\Delta_{2i} &= -\Delta_{12} (2a^i e^1 + b^i e^2) \quad \text{and} \\
D^i_{uu} &= 2\Delta_{12} (a_i (e_u^1)^2 + b_i e_u^1 e_u^2 + c_i (e_u^2)^2). 
\end{align*}
\]
\[
K = \sum_{i,j=3,4,5} \frac{P_{ij}}{\|\mathbf{p}_1 \times \mathbf{p}_2\|^4} \cdot f_{ij} \quad \text{and} \quad H = \sum_{i=3,4,5} \frac{\tilde{P}_{ikl}}{\|\mathbf{p}_1 \times \mathbf{p}_2\|^3} \cdot \tilde{f}_{kl}
\]

with constant (!)

\[
f_{ij} = \begin{cases} 
4(a_i c_j + a_j c_i) - 2b_i b_j & \text{for } i \neq j \\
4a_i c_i - (b_i)^2 & \text{for } i = j
\end{cases} \quad \text{and} \quad \tilde{f}_{kl} = \begin{cases} 
c_i & \text{for } k = l = 1 \\
a_i & \text{for } k = l = 2 \\
-b_i/2 & \text{for } k \neq l
\end{cases}
\]
**Prautzsch’s algorithm** (Free-form splines)

- $v_1$ and $v_2$ eigenvectors to the subdominant eigenvalue $\lambda$ of the Catmull-Clark algorithm. (Then $e^1$ and $e^2$ have bi-degree 3.)

- Set $e^3 = (e^1)^2$, $e^4 = e^1 e^2$ and $e^5 = (e^2)^2$ with control nets $w_i, i = 3, 4, 5$. $w_1$ and $w_2$ are the control nets of $e^1$ and $e^2$, respectively, in a degree-doubled representation.

- Subdivision matrix $A = MDM^+$ where

  \[
  M := [\mathbf{1}, w_1, w_2, w_3, w_4, w_5], \quad D := \text{diag}(1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2), \quad M^+ := (M^t M)^{-1} M^t.
  \]

The only non-zero eigenvalues of $A$ are $1, \lambda (2\text{-fold}), \lambda^2 (3\text{-fold})$ corresponding to the eigenvectors $\mathbf{1}, w_1, \ldots, w_5$. 
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For what choices of eigenfunctions $e_1^1$ and $e_2^2$ of a GS is

**total degree** $\deg(\Delta_{12}) \leq 2 \deg(x_0) - 2$

**bi-degree** $\deg(\Delta_{12}) \leq 2 \deg(x_0) - 1$?
Define the tensor-product mapping of the subeigenfunctions \( E : (e^1, e^2) \) so that \( \text{deg}(E) = \text{bi-4} \) and \( \text{deg}(\Delta(e^1, e^2)) = \text{deg}(e^1_u e^2_v - e^2_u e^1_v) = 5. \)
Partial Answer: Construction

1. Choose \( e^1 \) and \( e^2 \) of the 4n corner patches initially to form \( q \) of true degree bi-2.
2. Choose the non-corner patches to be of true degree bi-3 and so that the ring of patches is \( C^2 \).
3. **Perturb** the \( x \)-component of the common coefficient of the corner patch. (no influence on next rings; \( \Delta_{12}(q_x + x, q_y + 0) \)).

Then \( \deg(E) = 5 \) for the non-corner patches and for the corner patch

\[
\deg(\Delta(e^1, e^2)) = \max\{(3, 3), (0, 0), (\max\{(3, 4) + (2, 1), (4, 3) + (1, 2)\}, (0, 0)\} = \text{bi-5}.
\]
Find the $e^3$, $e^4$, $e^5$ (solve the PDEs for their coefficients). Any volunteers?

Fits nicely with *alternative answer*:

New $C^2$ biquartic free-form surface splines (modification of my Oberwolfach construction 1998)
**Conclusion**

express curvatures of $m$th spline ring converging towards the EOP

\[
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$\mu/\lambda^2$: implies necessary constraints  
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It is worth looking for curvature continuous subdivision schemes  
whose regular rings are polynomial of degree less than 6!