Curvature of subdivision surfaces

— a differential geometric analysis and literature review —

Jörg Peters, jorg@cise.ufl.edu
Georg Umlauf, georg.umlauf@gmx.de

Motivation

Almost all subdivision algorithms in the current literature achieve *tangent continuity but not curvature continuity*.
(C^1 with infinite curvature!)

Motivation

Almost all subdivision algorithms in the current literature achieve *tangent continuity but not curvature continuity*.
(C^1 with infinite curvature!)

Why is it difficult to achieve curvature continuity at an extraordinary point (*EOP*)?

Motivation

Almost all subdivision algorithms in the current literature achieve *tangent continuity but not curvature continuity*.
(C^1 with infinite curvature!)

Why is it difficult to achieve curvature continuity at an extraordinary point (*EOP*)?

The quantities to measure are *Gaussian and mean curvature* in a neighborhood of an EOP!

Motivation

Almost all subdivision algorithms in the current literature achieve *tangent continuity but not curvature continuity*.
(C^1 with infinite curvature!)

Why is it difficult to achieve curvature continuity at an extraordinary point (*EOP*)?

The quantities to measure are *Gaussian and mean curvature* in a neighborhood of an EOP!

Sample result:
At EOP the determinant of the *Jacobian of the subdominant* eigenfunctions of a curvature continuous subdivision algorithm must have *lower degree* than the determinant of the Jacobian of the surface.

Motivation: Review

Understand important *lower bound* results better:
Sabin 91, ($\geq$*bi-4*)
Reif 93,96, ($\geq$*bi-6*)
Prauzsch,Reif 99, ($\geq$*bi*-$r(k+1)$)
(Lower bounds on parametrization, not surface)

Understand *constructions* of curvature continuous piecewise polynomial subdivision algorithms
Prautzsch 97,
Prautzsch, Umlauf 98, Umlauf 99 (hybrid)
Reif 98.

Understand *stiffness* of such subdivision algorithms:
infinite collection of polynomial pieces
but generated by the *same* rule.

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
μ/λ^2: implies necessary constraints
Necessary and sufficient contraints: *PDEs*

- Lower bounds
- Prautzsch's sufficient condition and construction.
- The key open problem! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: PDEs

- Lower bounds
- Prautzsch's sufficient condition and construction.
- The key open problem! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: PDEs
- Lower bounds
- Prautzsch's *sufficient condition and construction*.
- The key open problem! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: PDEs

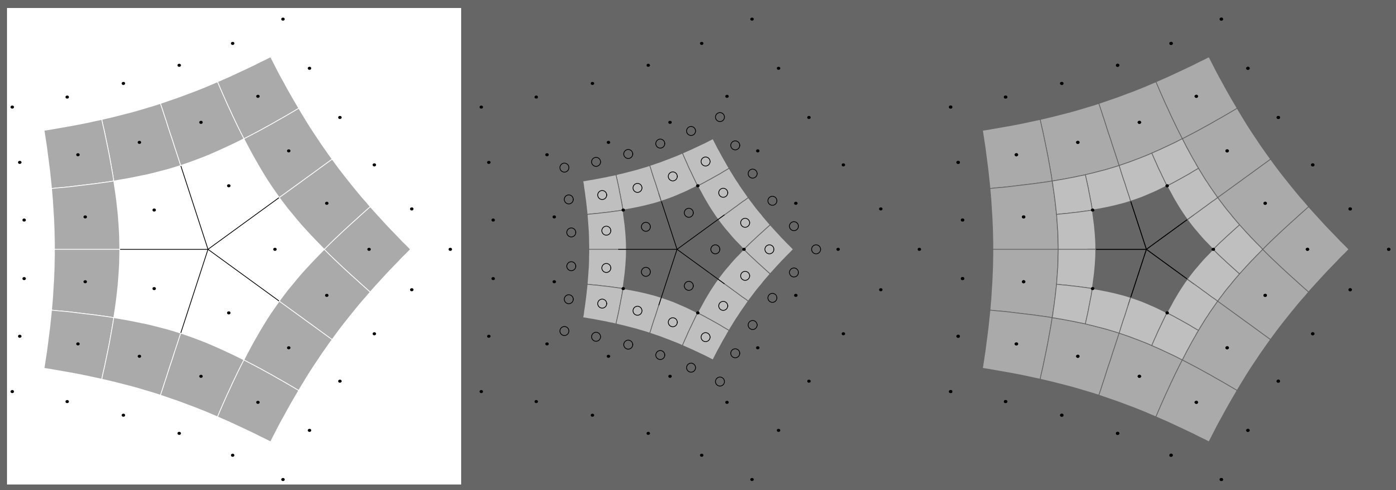
- Lower bounds
- Prautzsch's sufficient condition and construction.
- The *key open problem*! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Setting and definitions

The talk focusses generic subdivision (*GS*):
generalization of C^2 box-spline subdivision generating regular C^1 surfaces;
affine invariant, symmetric, linear, local, stationary.
However applies to non-generic cases [Reif 98 (habil), Zorin 98 (thesis)] and non-polynomial cases.

Surface rings are box-splines (with basis $\mathcal{B}(u,v)$)

$$\mathbf{x}_m : \{0,\ldots,n-1\} \times \Omega \to R^3, \qquad \mathbf{x}_m(u,v) = \mathcal{B}(u,v)\mathbb{C}_m,$$



Setting and definitions

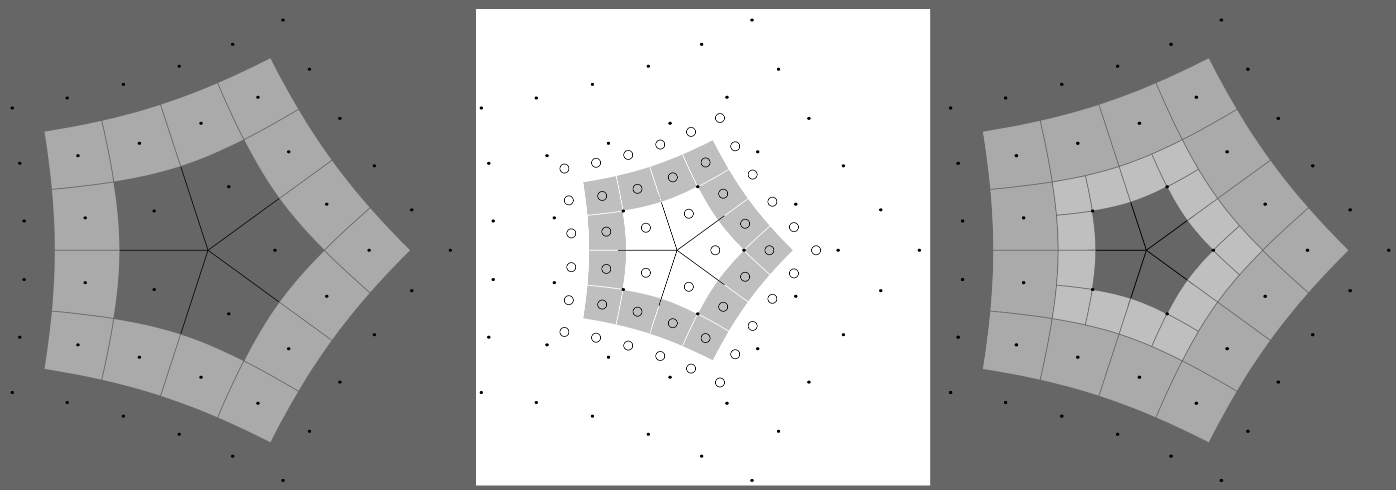
A is square, stochastic *subdivision matrix*: $\mathbb{C}_m = A^m \mathbb{C}_0$, diagonalizable with eigenvalues

$$1 = \lambda_0 > \underbrace{\lambda_1 = \lambda_2}_{=:\lambda} > \underbrace{\lambda_3 = \lambda_4 = \lambda_5}_{=:\mu} > \cdots \geq 0,$$

where $\lambda_1 = \lambda_2$ correspond to the 1st and $(n-1)$st block, $\lambda_3 = \lambda_4$ (for $n > 3$) to the 2nd and $(n-2)$nd block and λ_5 to the 0th block of the Fourier decomposition of A.

$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for all i yields eigendecomposition

$$\mathbb{C}_m = \sum_i \lambda_i^m \mathbf{v}_i \mathbf{p}_i, \qquad \mathbf{p}_i \in R^3.$$



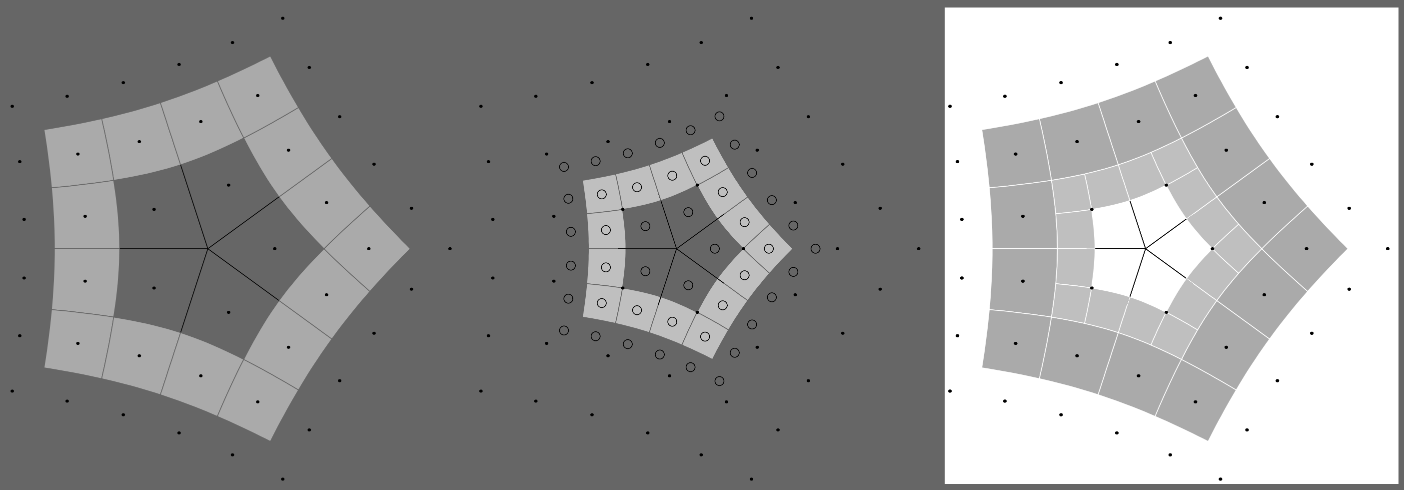
Setting and definitions

Expanded in the *eigenfunction*

$$\mathbf{e}^i : \{0, \ldots, n-1\} \times \Omega \to R, (u, v) \mapsto \mathcal{B}(u, v)\mathbf{v}_i$$

the surface ring $\mathbf{x}_m$ is of the form

$$\mathbf{x}_m(u, v) = \sum_i \lambda_i^m \mathcal{B}(u, v)\mathbf{v}_i\mathbf{p}_i = \sum_i \lambda_i^m \mathbf{e}^i(u, v)\mathbf{p}_i.$$



Gauss curvature K and the mean curvature H are

$$K(u,v) = \frac{e(u,v)g(u,v) - f(u,v)^2}{E(u,v)G(u,v) - F(u,v)^2},$$
$$H(u,v) = \frac{e(u,v)G(u,v) - 2f(u,v)F(u,v) + g(u,v)E(u,v)}{2(E(u,v)G(u,v) - F(u,v)^2)},$$

$$E = \mathbf{x}_u\mathbf{x}_u^t, \quad F = \mathbf{x}_u\mathbf{x}_v^t, \quad G = \mathbf{x}_v\mathbf{x}_v^t,$$
$$e = \mathbf{n}\mathbf{x}_{uu}^t, \quad f = \mathbf{n}\mathbf{x}_{uv}^t, \quad g = \mathbf{n}\mathbf{x}_{vv}^t,$$

and $\mathbf{n} = (\mathbf{x}_u \times \mathbf{x}_v)/\|\mathbf{x}_u \times \mathbf{x}_v\|$ is the normal. Since $\mathbf{x}$ is regular, $EG - F^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2$ is nonzero and

$$K = \frac{\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv}) - \det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})^2}{\|\mathbf{x}_u \times \mathbf{x}_v\|^4},$$
$$H = \frac{\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu})(\mathbf{x}_v\mathbf{x}_v^t) - 2\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})(\mathbf{x}_u\mathbf{x}_v^t) + \det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{vv})(\mathbf{x}_u\mathbf{x}_u^t)}{2\|\mathbf{x}_u \times \mathbf{x}_v\|^3}.$$

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: *PDEs*

- Lower bounds
- Prautzsch's sufficient condition and construction.
- The key open problem! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Gauss curvature and mean curvature at EOP

Expand into eigenfunctions $\mathbf{e}^i$ as in [Reif 93]

$$
\begin{aligned}
\mathbf{x}_u &= \lambda^m \left(\mathbf{e}_u^1\mathbf{p}_1 + \mathbf{e}_u^2\mathbf{p}_2\right) + \mu^m \left(\mathbf{e}_u^3\mathbf{p}_3 + \mathbf{e}_u^4\mathbf{p}_4 + \mathbf{e}_u^5\mathbf{p}_5\right) + o(\mu^m), \\
\mathbf{x}_{uv} &= \lambda^m \left(\mathbf{e}_{uv}^1\mathbf{p}_1 + \mathbf{e}_{uv}^2\mathbf{p}_2\right) + \mu^m \left(\mathbf{e}_{uv}^3\mathbf{p}_3 + \mathbf{e}_{uv}^4\mathbf{p}_4 + \mathbf{e}_{uv}^5\mathbf{p}_5\right) + o(\mu^m). \\
\mathbf{x}_u \times \mathbf{x}_v &= \lambda^{2m}\Delta_{12}(\mathbf{p}_1 \times \mathbf{p}_2) + o(\lambda^{2m}), \\
\det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu}) &= \lambda^{2m}\mu^m \sum_{i=3,4,5} \det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_i) D_{uu}^i + o(\lambda^{2m}\mu^m).
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{ij} &:= \mathbf{e}_u^i\mathbf{e}_v^j - \mathbf{e}_u^j\mathbf{e}_v^i, \\
D_{st}^i &:= \Delta_{12}\mathbf{e}_{st}^i - \Delta_{1i}\mathbf{e}_{st}^2 + \Delta_{2i}\mathbf{e}_{st}^1, \quad s,t \in \{u,v\}, \\
P_{ij} &:= \det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_i)\det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_j),
\end{aligned}
$$

$$K_m = \left(\frac{\mu}{\lambda^2}\right)^{2m} \frac{\sum_{i,j=3,4,5} P_{ij}\left(D^i_{uu} D^j_{vv} - D^i_{uv} D^j_{uv}\right) + o(1)}{\Delta_{12}^4 \|\mathbf{p}_1 \times \mathbf{p}_2\|^4 + o(1)}.$$

Δ_{12} is the Jacobi determinant of the subeigenfunctions (‘characteristic map’).
$\|\mathbf{p}_1 \times \mathbf{p}_2\|$ is positive for almost all initial control nets $\mathbb{C}_0$. Hence denominator ok.

- If $\mu > \lambda^2$ then the Gauss curvature at the EOP is infinite. [Catmull-Clark 78, Loop 87, Qu 90]

- If $\mu < \lambda^2$ then the Gauss curvature at the EOP is zero. [Prautzsch & Umlauf ’98]

- If $\mu = \lambda^2$ then the Gauss curvature at the EOP is bounded by the second factor of K_m but is possibly non-unique [Sabin 91,Holt 96].

Note combination of tangent continuity and infinite curvature for $\mu > \lambda^2$.

If $\mu = \lambda^2$ then the limit for $m \to \infty$ yields at the EOP

$$K = \sum_{i,j=3,4,5} \frac{P_{ij}}{\|\mathbf{p}_1 \times \mathbf{p}_2\|^4} \frac{D^i_{uu} D^j_{vv} - D^i_{uv} D^j_{uv}}{\Delta^4_{12}}.$$

a rational function in u and v that *must be constant*!

$P_{ij}(= P_{ji}) = \det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_i) \det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_j)$ arbitrary implies each summand has to be constant!

Eigenfunctions $\mathbf{e}^1, \ldots, \mathbf{e}^5$ must satisfy the *six partial differential equations* (G-PDE):

$$\begin{array}{rcll} D^i_{uu} D^j_{vv} - 2 D^i_{uv} D^j_{uv} + D^i_{vv} D^j_{uu} & = & \Delta^4_{12} \cdot \text{const}_{ij}, & \text{for } i, j \in \{3,4,5\},\ j > i, \\ D^i_{uu} D^i_{vv} - (D^i_{uv})^2 & = & \Delta^4_{12} \cdot \text{const}_{ii}, & \text{for } i = 3,4,5. \end{array}$$

Summary A GS has for almost all initial nets non-zero *Gauss curvature* at the EOP if and only if $\mu = \lambda^2$ and G-PDE holds. (9 additional partial differential equations for H)

General: GS is *curvature continuous* if $\mu = \lambda^2$ and the differential equations for G and H hold, because
the *principal curvatures*

$$\kappa_{1,2}^{m} = H_m \pm \sqrt{H_m^2 - K_m},$$

converge like $O(\mu^m/\lambda^{2m})$ for $m \to \infty$.

Since $\int d\mathbf{x}_m = O(\lambda^{2m})$ and $\mu < \lambda$

$$\sum_m \int_{\mathbf{x}_m} |\kappa_{1,2}^m|^2 d\mathbf{x}_m = \sum_m O(\mu^{2m}/\lambda^{2m}) < \infty.$$

which implies [Reif Schröder '00] for $p = 2$: The principal curvatures of the limit surface of a GS are *square integrable*.

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: *PDEs*

- Lower bounds
- Prautzsch's sufficient condition and construction.
- The key open problem! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Lower bounds on the degree

formal degree vs *true degree* $\quad \deg$ (= number of non-constant derivatives)

Recall Gauss PDE

$$\begin{aligned} D^i_{uu}D^j_{vv} - 2D^i_{uv}D^j_{uv} + D^i_{vv}D^j_{uu} &= \Delta^4_{12} \cdot \text{const}_{ij}, && \text{for } i, j \in \{3,4,5\},\ j > i, \\ D^i_{uu}D^i_{vv} - (D^i_{uv})^2 &= \Delta^4_{12} \cdot \text{const}_{ii}, && \text{for } i = 3,4,5. \end{aligned}$$

Simple count with $d = \deg(\mathbf{x}_0)$ total degree (resp. bi-degree) of regular parametrization. *Left side* of PDE

— total degree $\leq 2(2(d-1)+d-2) = 6d-8$

— bi-degree $\leq 2(2d-1+d-1) = 6d-4$,

whereas *right side* of PDE

— formal total degree of Δ^4_{12} is $4(2d-2)$

— formal bi-degree of Δ^4_{12} is $4(2d-1)$.

Degree mismatch: (unless $d = 0$)

If the *true* degree equals the *formal* degree

then GS is curvature continuous if and only if $\mu < \lambda^2$,

i.e. EOP is a flat point.

A GS with $\mu = \lambda^2$ is curvature continuous
only if the *true degree of the Jacobian Δ_{12} is less than its formal degree!*
Options:

(i) The true degree of $\mathbf{e}^1$ or $\mathbf{e}^2$ is less than d.

(ii) The leading terms in the Jacobian Δ_{12} cancel.

If not (ii) and not flat then $d' := \deg(\mathbf{e}^1) = \deg(\mathbf{e}^2)$, $d := \deg(\mathbf{x}_0)$):
total degree $\deg(\text{left}_{ij}) = 2(2d' + d - 4)$ and $\deg(\Delta_{12}^4) = 4(2d' - 2)$
bi-degree $\deg(\text{left}_{ij}) = 2(2d' + d - 2)$ and $\deg(\Delta_{12}^4) = 4(2d' - 1)$.
Compare to find $2d' = d$:

If *not (ii)* then GS is curvature continuous and not flat only if the true (bi-)degree of the surface is *at least twice* the true (bi-)degree of the subdominant eigenfunctions $\mathbf{e}^1$ and $\mathbf{e}^2$.

Comparison with earlier estimates

$2d' = d$ is consistent with *degree estimate of Reif 93, 96, Zorin 97*:

View surface as a function over the tangent plane parametrized by $\mathbf{e}^1$ and $\mathbf{e}^2$. Then non-flat implies non-tangential component at least quadratic in $\mathbf{e}^1$ and $\mathbf{e}^2$, i.e. $d \geq 2d'$.

[Prautzsch, Reif 99]
If the non-tangential component of the surface is at least of degree r in $\mathbf{e}^1$ and $\mathbf{e}^2$ then the surface representation has to be at least of degree rd'.
Since $\mathbf{e}^1$ and $\mathbf{e}^2$ have to have a minimal degree to form C^k rings, e.g. $d' \geq k+1$ in the tensor-product case, a lower bound is $r(k+1)$.

(parametrization dependent reasoning about surfaces!)

Or – (i) the leading terms of Δ_{12} cancel

- total degree: $\deg(\text{left}_{ij}) = 2\max\{\deg(\Delta_{12}) + d - 2, 2(d-1) + d - 2\} = 6d - 8$
- bi-degree: $\deg(\text{left}_{ij}) = 2\max\{\deg(\Delta_{12}) + d - 1, 2d - 1 + d - 1\} = 6d - 4$.

Comparing with $\deg(\Delta_{12}^4) = 4\deg(\Delta_{12})$.

If the true degree of $\mathbf{e}^1$ and $\mathbf{e}^2$ is not less than d
then GS is curvature continuous and not flat only if
the total degree $\deg(\Delta_{12}) \leq 3d/2 - 2$, (*bi-degree* $\deg(\Delta_{12}) \leq 3d/2 - 1$).

That is possible! E.g. if bi-$d = 4$ then $\deg(\Delta_{12}) = 5$ is needed
as if $\deg(\mathbf{e}^1) = \deg(\mathbf{e}^2) = 3$

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: PDEs

- Lower bounds
- Prautzsch's *sufficient condition and construction*.
- The key open problem! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Curvature continuous subdivision constructions

[Prautzsch & Umlauf ’98]:
induce *flat spots* to get low degree, small mask, curvature continuous subdivision algorithms.

[Sabin 91, Holt 96]:
adapt the leading eigenvalues to get non-zero bounded curvature.

Otherwise need *degree-reduced Jacobian*.

(Trivial) regular case of any C^2 box-spline: $\mathbf{e}^1$ and $\mathbf{e}^2$ are linear.

(Non-trivial) Projection of Prautzsch ’97, Reif ’98.

Prautzsch 98: *Sufficient conditions*

$$\mathbf{e}^i = a_i(\mathbf{e}^1)^2 + b_i\mathbf{e}^1\mathbf{e}^2 + c_i(\mathbf{e}^2)^2, \quad a_i, b_i, c_i \in R, \quad \text{for } i = 3, 4, 5.$$

Then (proof)

$$\begin{aligned}
\mathbf{e}^i_u &= 2a^i\mathbf{e}^1\mathbf{e}^1_u + b^i(\mathbf{e}^1_u\mathbf{e}^2 + \mathbf{e}^1\mathbf{e}^2_u) + 2c^i\mathbf{e}^2\mathbf{e}^2_u,\\
\mathbf{e}^i_{uu} &= 2a^i\left((\mathbf{e}^1_u)^2 + \mathbf{e}^1\mathbf{e}^1_{uu}\right) + b^i\left(\mathbf{e}^1_{uu}\mathbf{e}^2 + 2\mathbf{e}^1_u\mathbf{e}^2_u + \mathbf{e}^1\mathbf{e}^2_{uu}\right) + 2c^i\left((\mathbf{e}^2_u)^2 + \mathbf{e}^2\mathbf{e}^2_{uu}\right)\\
\Delta_{1i} &= \Delta_{12}(b^i\mathbf{e}^1 + 2c^i\mathbf{e}^2),\\
\Delta_{2i} &= -\Delta_{12}(2a^i\mathbf{e}^1 + b^i\mathbf{e}^2) \qquad \text{and}\\
D^i_{uu} &= 2\Delta_{12}(a_i(\mathbf{e}^1_u)^2 + b_i\mathbf{e}^1_u\mathbf{e}^2_u + c_i(\mathbf{e}^2_u)^2).
\end{aligned}$$

$$K = \sum_{i,j=3,4,5} \frac{P_{ij}}{\|\mathbf{p}_1 \times \mathbf{p}_2\|^4} \cdot f_{ij} \qquad \text{and} \qquad H = \sum_{\substack{i=3,4,5 \\ k,l=1,2,\ k \geq l}} \frac{\tilde{P}_{ikl}}{\|\mathbf{p}_1 \times \mathbf{p}_2\|^3} \cdot \tilde{f}_{kl}$$

with *constant* (!)

$$f_{ij} = \begin{cases} 4(a_i c_j + a_j c_i) - 2 b_i b_j & \text{for } i \neq j \\ 4 a_i c_i - (b_i)^2 & \text{for } i = j \end{cases}, \qquad \tilde{f}_{kl} = \begin{cases} c_i & \text{for } k = l = 1 \\ a_i & \text{for } k = l = 2 \\ -b_i/2 & \text{for } k \neq l \end{cases}.$$

Prautzsch's algorithm (Free-form splines)

- $\mathbf{v}_1$ and $\mathbf{v}_2$ eigenvectors to the subdominant eigenvalue λ of the Catmull-Clark algorithm. (Then $\mathbf{e}^1$ and $\mathbf{e}^2$ have bi-degree 3.)

- Set $\mathbf{e}^3 = (\mathbf{e}^1)^2, \mathbf{e}^4 = \mathbf{e}^1\mathbf{e}^2$ and $\mathbf{e}^5 = (\mathbf{e}^2)^2$ with control nets $\mathbf{w}_i, i = 3, 4, 5$. $\mathbf{w}_1$ and $\mathbf{w}_2$ are the control nets of $\mathbf{e}^1$ and $\mathbf{e}^2$, respectively, in a *degree-doubled representation*.

- Subdivision matrix $A = MDM^+$ where

$$M := [\mathbf{1}, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5],\ D := \mathrm{diag}(1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2),\ M^+ := (M^tM)^{-1}M^t.$$

The only non-zero eigenvalues of A are $1, \lambda(2\text{-fold}), \lambda^2(3\text{-fold})$ corresponding to the eigenvectors $\mathbf{1}, \mathbf{w}_1, \ldots, \mathbf{w}_5$.

Talk Outline

- The (few) basics. (nomenclature)
- express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

  for scalar constants $\mu < \lambda$ and rational functions f_K, f_H.
  μ/λ^2: implies necessary constraints
  Necessary and sufficient contraints: PDEs
- Lower bounds
- Prautzsch's sufficient condition and construction.
- The *key open problem*! (well, sort of)
- preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

Big Question

For what choices of eigenfunctions $\mathbf{e}^1$ and $\mathbf{e}^2$ of a GS is

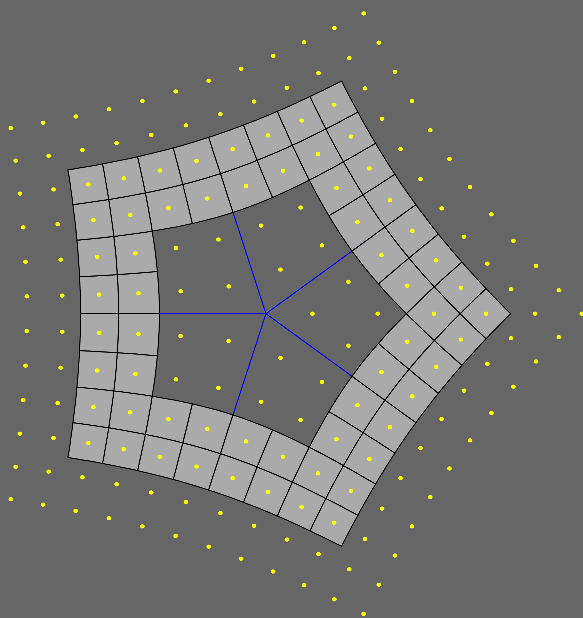
total degree $\deg(\Delta_{12}) \leq 2\deg(\mathbf{x}_0) - 2$

bi-degree $\deg(\Delta_{12}) \leq 2\deg(\mathbf{x}_0) - 1$?

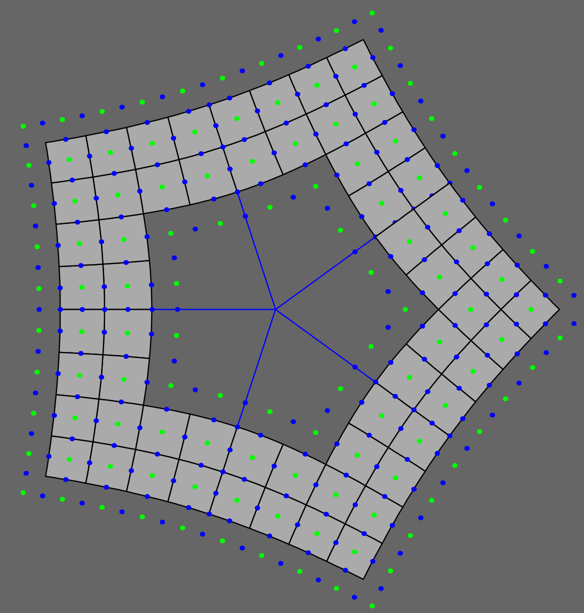
Partial Answer

Define the tensor-product mapping of the subeigenfunctions $E : (\mathbf{e}^1, \mathbf{e}^2)$ so that $\deg(E) =$ *bi-4* and $\deg(\Delta(\mathbf{e}^1, \mathbf{e}^2)) = \deg(\mathbf{e}^1_u \mathbf{e}^2_v - \mathbf{e}^2_u \mathbf{e}^1_v) = 5$.

C^2 quartics:

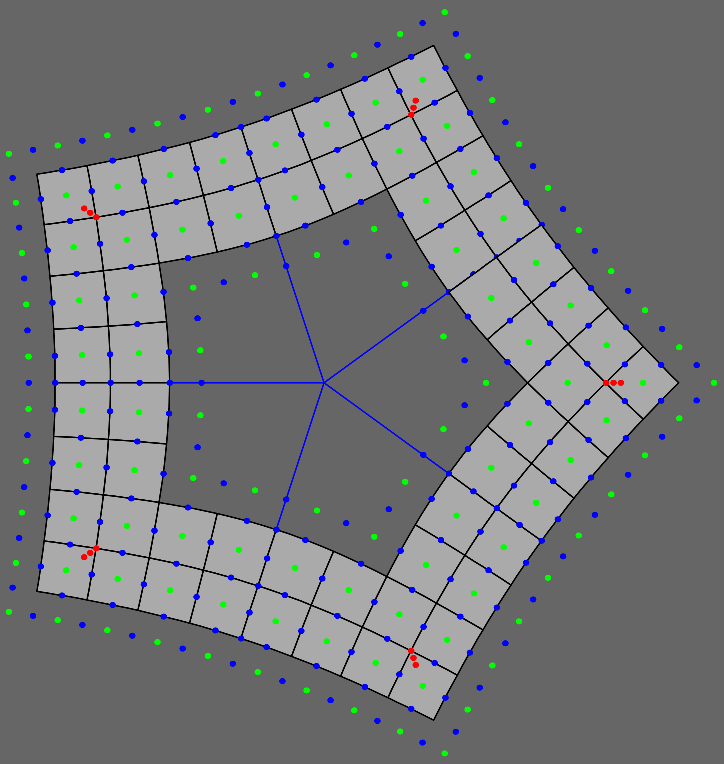


knot insertion $\rightarrow$



Partial Answer: Construction

1. Choose $\mathbf{e}^1$ and $\mathbf{e}^2$ of the $4n$ *corner patches* initially to form q of true degree *bi-2*.
2. Choose the non-corner patches to be of true degree bi-3 and so that the ring of patches is C^2.
3. *Perturb* the x-component of the common coefficient of the corner patch. (no influence on next rings; $\Delta_{12}(q_x + x, q_y + 0)$).



Then $\deg(E) = 5$ for the non-corner patches and for the corner patch

$$\deg(\Delta(\mathbf{e}^1, \mathbf{e}^2)) = \max\{(3,3), (0,0), (\max\{(3,4) + (2,1), (4,3) + (1,2)\}, (0,0)\}$$
$$= \text{bi-}5.$$

Find the $\mathbf{e}^3$, $\mathbf{e}^4$, $\mathbf{e}^5$ (solve the PDEs for their coefficients). Any volunteers?

Fits nicely with *alternative answer:*

New C^2 biquartic free-form surface splines (modification of my Oberwolfach construction 1998)

Conclusion

express curvatures of mth spline ring converging towards the EOP

$$K_m = (\mu/\lambda^2)^{2m} f_K^m(u,v), \quad H_m = (\mu/\lambda^2)^m f_H^m(u,v)$$

μ/λ^2: implies necessary constraints
Necessary and sufficient contraints: PDEs
Lower bounds
Prautzsch's sufficient condition and construction.
The key open problem
preprint: http://www.cise.ufl.edu/research/SurfLab/papers/

It is worth looking for curvature continuous subdivision schemes

whose regular rings are polynomial of degree less than 6!