Smooth Patching of Refined Triangulations

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Abstract

This paper presents a simple algorithm for associating a smooth, low degree polynomial surface with triangulations whose extraordinary mesh nodes are separated by sufficiently many ordinary, 6-valent mesh nodes. Output surfaces are at least tangent continuous and are C^2 sufficiently far away from extraordinary mesh nodes; they consist of three-sided Bézier patches of degree 4. In particular, the algorithm can be used to skin a mesh generated by a few steps of Loop's generalization of three-direction box-spline subdivision.

1 Overview

Three-direction subdivision [15] enriched by Loop's averaging rule for extraordinary mesh nodes [8] is a natural refinement strategy for smoothing triangulated data sets. Due to the underlying box-spline, the limit surface consists largely of degree 4, three-sided polynomial pieces that join C^2. Near extraordinary mesh nodes, i.e. nodes of the triangulation that have more or less than the regular number of $\mathtt{n} = 6$ neighbors, the surface consists of an *infinite sequence* of ever-smaller nested rings of degree 4, three-sided polynomial pieces (c.f. Figure 1). Subsequent rings contribute less to the surface while the cost stays constant (respectively grows exponentially if the refinement is applied everywhere). The resulting surface pieces are less and less desirable since their curvature increases without bound [4, 10].

Rather than generating ever finer rings, the algorithm presented here smoothly fills n-sided holes corresponding to extraordinary mesh nodes in one step using n degree 4, three-sided polynomial pieces. This yields, for example, surfaces that agree with the Loop limit surface everywhere except in the polynomial pieces attached to the extraordinary points. The position of the extraordinary points and their normals can be chosen freely, in particular so as to match the position and normal of the Loop limit surface.

The algorithm applies to triangulations with sufficiently separated extraordinary mesh nodes, as is typically the case in semiregular triangulations. One way to achieve separation is to apply the 1 to 4 split underlying Loop subdivision. However, other geometric rules than Loop's for refining the triangulation can be used, e.g. defining

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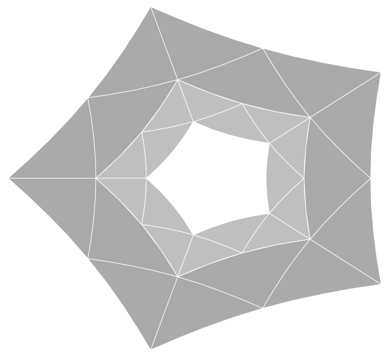

Figure 1: Two nested rings of $3n$ 3-sided patches contracting towards an extraordinary point of valence $n = 5$.

node positions by fitting data, by other averaging strategies [17, 6, 7], or to distribute curvature and to blend regions with varying sharpness [12, 5, 16]. In all cases regular submeshes can be interpreted as box-spline meshes and extraordinary mesh nodes are isolated.

The option of pulling the nodes of the triangulation to create regions of high or of low curvature, e.g. blends or semi-smooth creases, or to match data is the major reason for performing one or two steps of subdivision on a coarse triangulation rather than fitting the surface directly with one degree 6 piece per triangle as proposed in [9]. While it is theoretically possible to apply the construction to a large, irregular triangulation, a typical use would be to apply it to a low resolution model and take advantage of the higher approximation order than flat triangles that degree 4 Bézier patches offer.

The main benefit of applying the algorithm is that the resulting smooth free-form surface can be exactly represented in a fixed, standard format, namely Bézier pieces of the minimal possible degree 4 that captures the degree of the three-direction subdivision surface. (To obtain smooth free-form surfaces in general already degree 3 suffices [12].) The curved triangle pieces can then be sent individually to the graphics chip for adaptive evaluation.

An analogous construction for smoothly patching refined quadrilateral meshes is described in [14]. [14] similarly applies to data fitted meshes and to meshes generated by the Catmull-Clark generalization of bicubic Nurbs subdivision.

2 Three-direction Box-spline and Bézier Patches

In his 1976 thesis [15], Sabin showed how to construct C^2 box-splines for a partition of the plane into equilateral triangles (see also [2, 1]) and gave a Bézier representation of the polynomial pieces of this box-spline: a mesh node P_0 and its 6 direct neighbors $P_1(i)$ define four different types of coefficients of the degree 4 Bézier polynomials

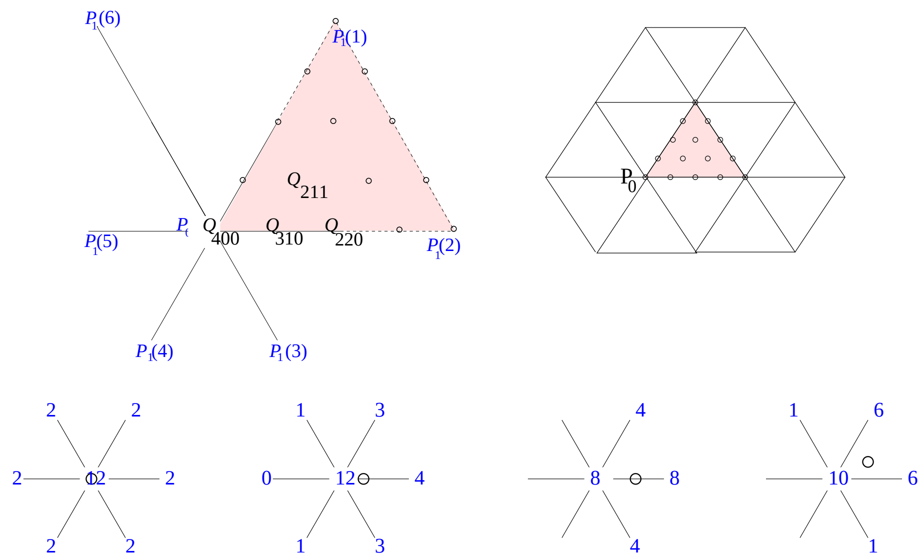

Figure 2: A mesh node P_0 is called regular if it has exactly 6 direct neighbors $P_1(i)$ (*top left*). By symmetry, there are only four distinct rules for determining the 15 coefficients Q_{ijk} of a degree 4 Bézier patch of the C^2 box-spline surface of a regular subtriangulation (*top right*). These rules are illustrated (*bottom*) with $\circ$ indicating the position of the Bézier coefficient relative to the vertices of the triangulation. The weights are scaled by 24.

$\sum_{i+j+k=4} Q_{ijk} \frac{4!}{i!j!k!} u^i v^j (1-u-v)^k$ (c.f. Figure 2) via

$$
\begin{aligned}
24Q_{400} &= 12P_0 + 2\sum_{i=1}^{6} P_1(i), \\
24Q_{310} &= 12P_0 + 3P_1(1) + 4P_1(2) + 3P_1(3) + P_1(4) + P_1(6), \\
24Q_{211} &= 10P_0 + 6P_1(1) + 6P_1(2) + P_1(6) + P_1(3), \\
24Q_{220} &= 8P_0 + 8P_1(2) + 4P_1(1) + 4P_1(3).
\end{aligned}
$$

Figure 2 *bottom* shows the formulas as stencils or masks; the remaining coefficients of a degree 4, three-sided Bézier patch are obtained by symmetry. The planform or footprint, i.e. the subtriangulation needed to define all the Bézier coefficients of a patch, is shown in Figure 2 *top right*.

3 From Mesh to Surface at Extraordinary Nodes

The previous section explained how to convert a regular mesh to a C^2 surface. This leaves only the neighborhood of extraordinary points undefined. Consider two rings

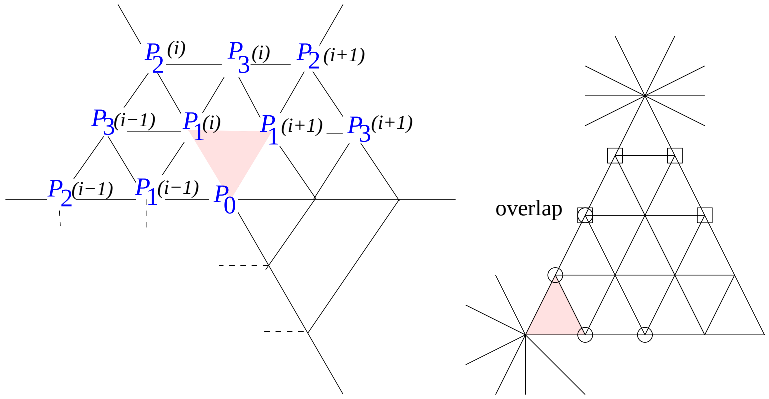

Figure 3: (*left*) Position of the mesh nodes $P_k(i)$ in the neighborhood of an extraordinary mesh node P_0. (*right*) Potential overlap of P_2 nodes of two adjacent nodes of even valence. The P_2 nodes may have to be perturbed to meet the even–alternation requirement $\sum_{l=1}^{\mathtt{n}}(-1)^l P_2(l) = 0$.

of nodes $P_1(i)$, respectively $P_2(i)$ and $P_3(i)$, with $P_k(i) \in \mathbf{R}^3$, $k \in \{1,2,3\}$ and $i \in \{1,\ldots,\mathtt{n}\}$ that surround an $\mathtt{n}$-valent extraordinary mesh node P_0 in clockwise order as shown in Figure 3. We collect the mesh nodes $P_k(l)$ into vectors of nodes $P_k \in \mathbf{R}^{\mathtt{n}\times 3}$ and the Bézier coefficient $Q_{ijk}(l)$ into vectors of coefficients $Q_{ijk} \in \mathbf{R}^{\mathtt{n}\times 3}$.

An outline of the conversion to a C^1 surface near extraordinary mesh nodes is as follows. Sabin's rules determine the $\mathtt{n}$ times 9 Bézier coefficients corresponding to the shaded areas in Figure 4: $Q_{i,j,k}(l) \in \mathbf{R}^3$ for $i < 2$ and $i+j+k = 4$, $l = 1,\ldots,\mathtt{n}$. The algorithm then allows a choice of the extraordinary point Q_{400} and its normal by prescribing Q_{310} and generates Q_{202}, Q_{211} and Q_{220} so that the polynomial pieces join smoothly after a perturbation of Q_{022}, Q_{121} and Q_{112}.

3.1 Input and output of the algorithm

The *input* to the algorithm are the position of the extraordinary mesh node P_0 and the vectors P_1, P_2 and P_3 of the two rings surrounding it. Optional inputs are the limit point Q_{400} and the normal by specifying P_1 and hence Q_{310}.

Even–alternation requirement: if $\mathtt{n}$ is even but not 6, two alternating sums of the radial neighbors of the extraordinary mesh node must be zero:

$$\mathtt{A}_k := \sum_{l=1}^{\mathtt{n}}(-1)^l P_k(l) = 0, \text{ for } k = 1,2.$$

That is, we force the average of the even-labeled neighbors to equal the average of the odd-labeled neighbors. The requirement is not needed if $\mathtt{n}$ is odd or $\mathtt{n} = 6$. If the P_k have been generated by the rules of the three-direction box spline or, equivalently, Loop's subdivision then $\mathtt{A}_1 = 0$ and $\mathtt{A}_2 \to \mathtt{A}_2/16$ with each subdivision step. Section

4 explains the technical reason for the requirement. If the rings of P_k nodes do not overlap the rewuirement can be met by computing $r = \mathtt{A}_k/\mathtt{n}$ and, if $r \neq 0$, adding $-(-1)^l r$ to $P_k(l)$ for $l = 1, \dots, \mathtt{n}$. This is an explicit solution to the underconstrained problem with $\mathtt{n}$ unknowns and one constraint and minimizes the perturbation of the mesh. If there is overlap (see Figure 3 *right*) we solve a larger underdetermined system. The technical report [13] lists explicit formulas for patches of degree 5 and degree 6 that join smoothly at the extraordinary point and do not need to meet the requirement.

The default choice of the extraordinary point and its normal are the limit position and normal of the extraordinary mesh node under subdivision with Loop's rule. According to ([8], p42) this limit position is

$$Q_{400} = \alpha P_0 + (1-\alpha)\sum_{i=1}^{n} P_{1,i}/n, \qquad \alpha = \frac{24}{55-12c-4c^2}, c = \cos(\frac{2\pi}{\mathtt{n}}).$$

I prefer $\alpha = (4+c)/9$ as suggested in [9]. The polynomial surface has the limit normal of the Loop subdivision surface if

$$Q_{310} = P_0 + \mathbf{A}_n P_1/4$$

for the n by n matrix $\mathbf{A}_n$ given below. The implied scaling of the tangent vectors $Q_{310}(i) - Q_{400}(i)$ is consistent with the regular Sabin conversion rules for $\mathtt{n} = 6$.

The *output* are n three-sided Bézier patches of degree 4 that join each other and the next ring of patches with tangent continuity. The patch corresponding to $P_1(l)$, $P_1(l+1)$, $P_3(l)$ is the lth *edge-adjacent patch*. Its coefficients are labeled $Q^{adj}_{ijk}(l)$ as indicated in Figure 4. For the implementation note that $Q^{adj}_{022}(l) = Q_{022}(l)$ changes as part of the output.

3.2 The algorithm

We define $Q^{-}_{ijk}(l) = Q_{ijk}(l-1)$ where $Q_{ijk}(0) = Q_{ijk}(\mathtt{n})$, and the $\mathtt{n}$ by $\mathtt{n}$ matrices $\mathbf{A}_\mathtt{n}$ and $\mathbf{B}_\mathtt{n}$ with rows $i = 1, \dots, \mathtt{n}$ and columns $j = 1, \dots, \mathtt{n}$ and entries

$$\mathbf{A}_\mathtt{n}(i,j) = \frac{2\beta}{\mathtt{n}}\cos(\frac{2\pi}{\mathtt{n}}(i-j)) \qquad \beta = 1\text{(default) and}$$

$$\mathbf{B}_\mathtt{n}(i,j) = \begin{cases} (-1)^{n_{i-j}} & \text{if } \mathtt{n} \text{ is odd,} \\ (-1)^j - 2n_{i-j}(-1)^{j-i}/\mathtt{n} & \text{if } \mathtt{n} \text{ is even,} \end{cases}$$

$$n_{i-j} = \mod(\mathtt{n}+i-j, \mathtt{n}).$$

For example, $\mathbf{B}_6(3,3) = -1$. Choosing $\beta > 1$ enlarges the $\mathtt{n}$-gon spanned by the $Q_{310}(i)$ and thereby decreases the curvature at the extraordinary point.

Then the coefficients of the lth patch that are fixed by Sabin's conversion rules, by

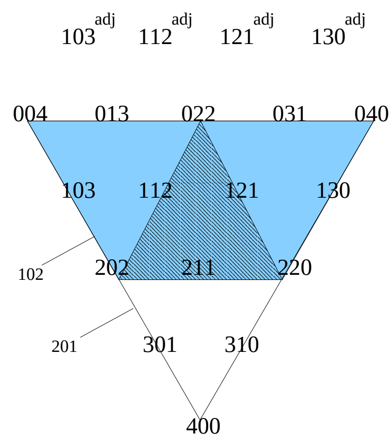

Figure 4: Indices of the Bézier coefficients (control points) of a degree 4 patch. If 400 is the index of an extraordinary point then the coefficients in the shaded area are initially determined by Sabin's rules. The coefficients corresponding to the darker, central area are subsequently modified. The indices for control points of the edge-ajacent patch have a superscript $^{\text{adj}}$ and $Q_{0ij} = Q^{\text{adj}}_{0ij}$.

the optional input and

$$
\begin{aligned}
&Q_{301} \leftarrow Q_{400} + \mathbf{A}_{\mathtt{n}} P_1/4, \\
&\quad q_{102} = (4Q_{103} - Q_{004})/3, \\
&\quad q_{201} = (4Q_{301} - Q_{400})/3, \\
&Q_{202} \leftarrow (q_{102} + q_{201})/2, \quad c = \cos(\frac{2\pi}{\mathtt{n}}) \\
&\quad r_{211} = Q_{301} + \frac{c}{2}(q_{102} - q_{201}) + \frac{1}{8}(q_{201} - Q_{400}), \\
&Q_{211} \leftarrow \mathbf{B}_n r_{211}, \\
&\quad r_{022} = Q_{202} + \frac{1}{2}(Q^{adj,-}_{121} - Q^{-}_{031} + Q^{adj}_{112} - Q_{013}) + \frac{c}{4}(Q_{004} - q_{102}) + \frac{1}{4}(q_{102} - q_{201}), \\
&Q_{022} \leftarrow \mathbf{B}_{\mathtt{n}} r_{022}, \\
&Q_{112} \leftarrow Q_{013} + Q_{022} - Q^{adj}_{112}, \\
&Q_{121} \leftarrow Q_{031} + Q_{022} - Q^{adj}_{121}, \\
&Q^{adj}_{022} \leftarrow Q_{022}.
\end{aligned}
$$

At global boundaries of the triangulation the Bézier coefficients are not constrained by smoothness and can match additional boundary conditions, e.g. position and tangency.

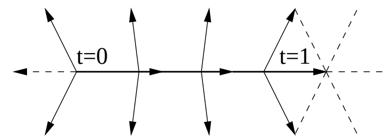

Figure 5: Each of the four Bézier (difference) coefficients of $\frac{\partial q_i}{\partial u_i}(0,t)$ (pointing *up*) plus $\frac{\partial q_{i+1}}{\partial v_{i+1}}(t,0)$ (pointing *down*) has to equal a linear combination of the three coefficients of $\frac{\partial q_i}{\partial v_i}(0,t)$ (*horizontal*).

4 Smoothness

We *claim*: the output patches join at least with tangent continuity across the edges of the three-sided patches attached to the extraordinary mesh node. For all other edges, the box-spline conversion guarantees a C^1 transition, respectively a C^2 transition if none of the Bézier coefficients are perturbed. This pattern of smoothness is illustrated in Figure 7.

Let u_i, v_i be the parameters of patch i and $v_i = u_{i+1} = t \in [0,1]$ the parameter along the boundary between patch q_i and patch q_{i+1}, and $c = \cos(\frac{2\pi}{\mathtt{n}})$. The algorithm enforces the polynomial equation

$$\Big(2c(1-t)+t\Big)\frac{\partial q_i}{\partial v_i}(0,t) = \frac{\partial q_i}{\partial u_i}(0,t) + \frac{\partial q_{i+1}}{\partial v_{i+1}}(t,0).$$

Here the degree 4 patches have boundary curves of degree 3 (with coefficients Q_{400}, q_{201}, q_{102}, Q_{004}) emanating from the extraordinary point. The equation is verified in the standard way by equating its four Bézier coefficients [11].

As illustrated in Figure 5 if the extraordinary mesh node has $\mathtt{n} = 3$ neighbors then $c = -0.5$ and $(2t-1)\partial q_i/\partial t = \partial q_i/\partial u_i + \partial q_{i+1}/\partial v_{i+1}$. In particular, at $t = 0$, $-\frac{\partial q_i}{\partial v_i} = \frac{\partial q_i}{\partial u_i}(0,t) + \frac{\partial q_{i+1}}{\partial v_{i+1}}(t,0)$ and at $t = 1$, $\frac{\partial q_i}{\partial v_i} = \frac{\partial q_i}{\partial u_i}(0,t) + \frac{\partial q_{i+1}}{\partial v_{i+1}}(t,0)$. The relation at $t = 0$ is enforced by applying the discrete Fourier convolution matrix $\mathbf{A}_n$. The second equation, for the mixed derivatives at $t = 0$, holds due to the choice of $\mathbf{B}_\mathtt{n}$ in computing Q_{211}; $\mathbf{B}_\mathtt{n}$ is the explicit inverse of the constraint matrix if $\mathtt{n}$ is odd and yields the least squares answer if $\mathtt{n}$ is even. This answer is a solution exactly if the alternating sum $\mathtt{A}_1 = 0$ (as one can check by expanding the contraint, via Sabin's rules, into expressions in terms of the mesh nodes). Recall, that $\mathtt{A}_1 = 0$ is *always* true if P_1 is the result of a three-direction subdivision step i.e. the edge rule of Loop's subdivision. Otherwise, the condition has to be enforced as explained earlier. In either case, the second equation now holds without error. The third equation holds by choice of $Q_{121}(i-1)$ and $Q_{112}(i)$. The fourth equation at the $t = 1$ holds due to the unchanged C^1 transition of the box spline. The C^1 continuity with the edge-adjacent patch is enforced by moving the boundary coefficient Q_{022}: again $\mathbf{B}_\mathtt{n}$ is the explicit inverse of the constraint matrix if $\mathtt{n}$ is odd and yields the least squares answer if $\mathtt{n}$ is even. Expansion shows that this answer is a solution if $\sum_{i=1}^{\mathtt{n}}(-1)^i P_2(i) = 0$ holds as enforced *a priori*.

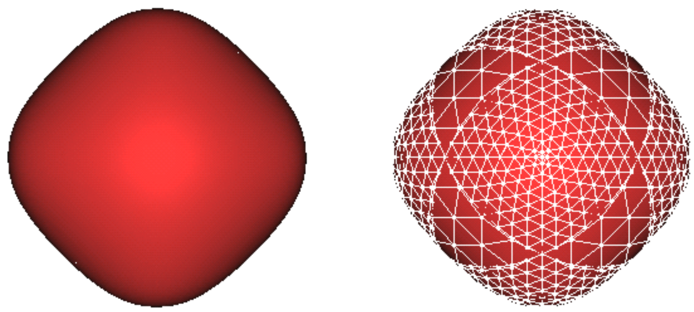

Figure 6: *Left:* Octahedral mesh smoothed after one Loop subdivision. *Right:* The same surface with a superimposed hierarchical Bézier coefficient mesh.

5 Summary

The algorithm presented in this paper gives simple, explicit formulas for converting refined triangulations to closed-form, smoothly-connected, standard Bézier patches of degree 4. These can be integrated with generalized three-direction box-spline subdivision to yield surfaces that differ from the limit surfaces of Loop subdivision only near the extraordinary mesh nodes (where the Bézier patches have finite curvature whereas the Loop limit surface generically has divergent curvature.) Analogous constructions using patches of degree 5 and 6 are given in the technical report [13].

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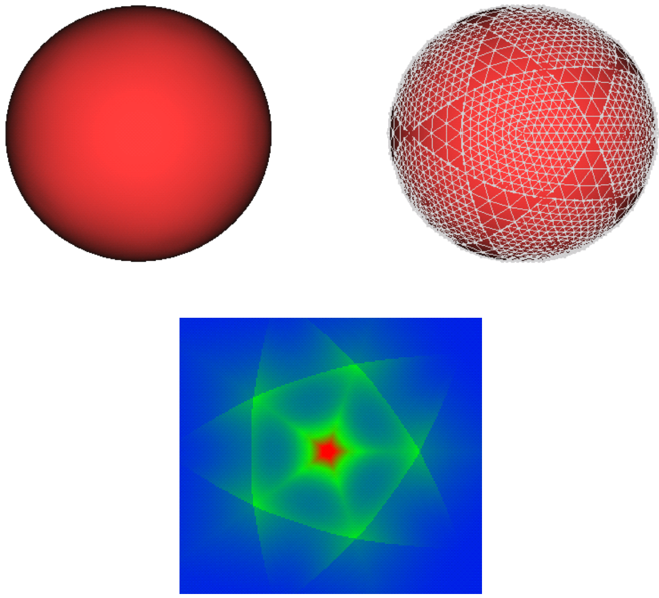

Figure 7: Icosahedral mesh (one Loop subdivision). *bottom:* Gauss curvature near the extraordinary point with curvature values mapped to colors so as to make the only tangent continuous transitions clearly visible. The maximal Gauss curvature for an input icosahedron of diameter $\sqrt{2}$ is 1.021.

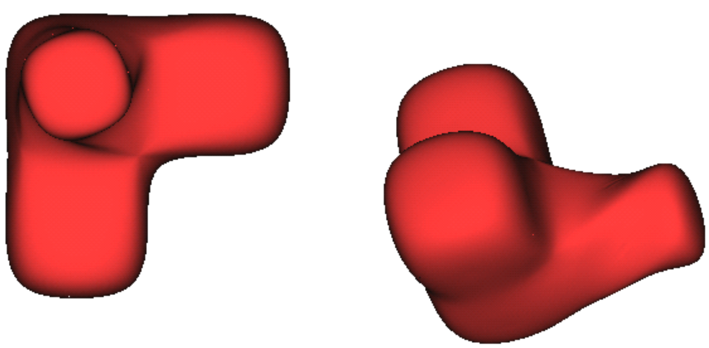

Figure 8: Two views of a twisted stencil (vertices of valence 4,6,8,10,12).

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