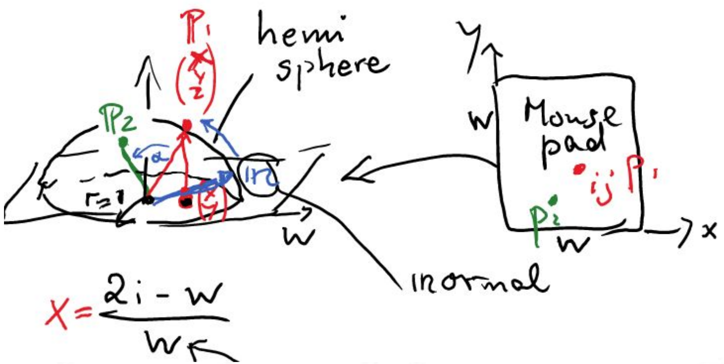
Trackball: 2D integer change of mouse position → 3D rotation

- Associate each mouse position $= (i, j)$ with the point on a unit hemisphere:
  (1) For w=width, scale
  the x component to $(2i - w)/w$ and
  the y component to $-(2j - w)/w$.
  (2) set $z := \sqrt{1 - x^2 - y^2}$.
  centered, but mouse starts at 0



- Two consecutive points P_1 and P_2 on the hemisphere define a normal direction $n = P_1 \times P_2$.

- Rotate about n.

Euler angles: rotation about axes

- Anisotropy, Coordinate system dependence:
  - Ordering and orientation of coordinate axes is important. (Parameters lack a simple, local geometric interpretation.)
- Finding the Euler angles for a given orientation is difficult
  - getMatrix () helps
- Gimbal Lock: a degree of freedom can vanish

- Easy to implement and widely used

Euler angles: Gimbal Lock

Mechanical problem:

gyroscopes using three nested rotating frames.

after a y-rotation by $\pi/2$ (rotates x-axis onto $-$z-axis)

an x-rotation by α equals z-rotation by $-\alpha$.

- Objects seem to ‘stick’.
- Some orientations are difficult to obtain from ‘the wrong direction’.
- interpolation through singularity behaves unpredictable.

Euler angles: Gimbal Lock

If $\alpha_y = \pi/2 \quad c_y = cos(\alpha_y)$ etc.

$$R_y = \begin{bmatrix} c_y & 0 & -s_y & 0 \\ 0 & 1 & 0 & 0 \\ s_y & 0 & c_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_x R_y R_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_x & -s_x & 0 \\ 0 & s_x & c_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_z & -s_z & 0 & 0 \\ s_z & c_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ s & c & 0 & 0 \\ c & -s & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$c = \cos(\alpha_x - \alpha_z)$ and $s = \sin(\alpha_x - \alpha_z)$

only one degree of freedom!

Quaternions: motivating analogy

2D rotation x = (x1 , x2) by angle α: $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \qquad \begin{array}{l} c := \cos(\alpha), \\ s := \sin(\alpha). \end{array}$

or, alternatively, expressing x as the complex number $x_1 + ix_2$ where $i := \sqrt{-1}$.
as multiplication in the complex field by

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

Quaternions: Definition and rules

$$\hat{\mathbf{q}} := q_0 + iq_1 + jq_2 + kq_3 = (q_0, q_1, q_2, q_3) =: (q_0, \mathbf{q}),$$

where $\mathbf{q} := (q_1, q_2, q_3)$ is a vector in R^3. Quaternions have the multiplication table

$\odot$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

This multiplication table is used to compute the product of two quaternions $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ which is again a quaternion:

$$\hat{\mathbf{p}} \odot \hat{\mathbf{q}} = (p_0 q_0 - (p_1 q_1 + p_2 q_2 + p_3 q_3), p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}).$$

We define $\|\hat{\mathbf{q}}\|^2 := q_0^2 + \mathbf{q} \cdot \mathbf{q}$. This implies (check that $\hat{\mathbf{q}}^{-1} \odot \hat{\mathbf{q}}$ is real)

$$\hat{\mathbf{q}}^{-1} = (q_0, -\mathbf{q}) / \|\hat{\mathbf{q}}\|^2$$

Quaternions: Construction and Use

Quaternion for rotation about $\mathbf{h} \in R^3$ by α:

$$\hat{\mathbf{q}} := (\cos(\alpha/2), \sin(\alpha/2)\frac{\mathbf{h}}{\|\mathbf{h}\|})$$

To rotate the point $\mathbf{v} := (v_1, v_2, v_3)$, we multiply the quaternions as

$$\hat{\mathbf{q}}^{-1} \odot (0, \mathbf{v}) \odot \hat{\mathbf{q}}. \tag{1}$$

Note: $\hat{\mathbf{q}}$ and $-\hat{\mathbf{q}}$ represent the same rotation and rotation by 0 and 360 degrees coincide.

Quaternions: Example

Example: Let $\mathbf{h} := (1,1,1)$, $\alpha := 2\pi/3$, $\mathbf{v} := (1,0,0)$. That is, we look along the diagonal axis $\mathbf{h}$ and see the three coordinate axes equally distributed with an angle of $2\pi/3$ between each pair. We therefore assume that the rotation will map the x-axis $\mathbf{v}$ to one of the other two axes. Since $\cos(\alpha/2) = 1/2$ and $\sin(\alpha/2) = \sqrt{3}/2$

$$\frac{\mathbf{h}}{\|\mathbf{h}\|} = (1,1,1)/\sqrt{3},$$

$$\hat{\mathbf{q}} = (\frac{1}{2}, \frac{\sqrt{3}}{2}\frac{(1,1,1)}{\sqrt{3}}) = \frac{1}{2}(1,1,1,1),$$

$$\|\hat{\mathbf{q}}\|^2 = (\frac{1}{2}^2 + \frac{1}{2}(1,1,1)\cdot\frac{1}{2}(1,1,1)) = 1,$$

$$\hat{\mathbf{q}}^{-1} = \frac{1}{2}(1,-1,-1,-1),$$

Quaternions: Example

Example: Let $\mathbf{h} := (1,1,1)$, $\alpha := 2\pi/3$, $\mathbf{v} := (1,0,0)$.

$$\begin{aligned}
\hat{\mathbf{q}}^{-1} &= \frac{1}{2}(1,-1,-1,-1), \\
\hat{\mathbf{q}}^{-1} \odot (0,\mathbf{v}) &= \left(0-(-\frac{1}{2}), \frac{1}{2}(1,0,0) + \frac{1}{2}(-1,-1,-1) \times (1,0,0)\right) \\
&= \left(\frac{1}{2}, (\frac{1}{2},0,0) - \frac{1}{2}(0,1,-1)\right) \\
&= \frac{1}{2}(1,1,-1,1)
\end{aligned}$$

Quaternions: Example

Example: Let $\mathbf{h} := (1,1,1)$, $\alpha := 2\pi/3$, $\mathbf{v} := (1,0,0)$.

$$\hat{\mathbf{q}}^{-1} \odot (0, \mathbf{v}) = \frac{1}{2}(1,1,-1,1) \qquad \hat{\mathbf{q}} = \frac{1}{2}(1,1,1,1),$$

$$\begin{aligned}\hat{\mathbf{q}}^{-1} \odot (0, \mathbf{v}) \odot \hat{\mathbf{q}} &= \frac{1}{2}\frac{1}{2}(1-1, (2,0,2) + (-2,0,2)) \\ &= (0,0,0,1).\end{aligned}$$

That is, we rotate the point on the x-axis onto the z-axis. (note that we look from the origin and rotate ccw; when looked at from $(1,1,1)$) the angle is clockwise)

Note: one can convert from Euler to Quaternion and from Quaternion to Euler
Relative rotation: roll,pitch yaw angle-based: azimuth, elevation

Quaternions