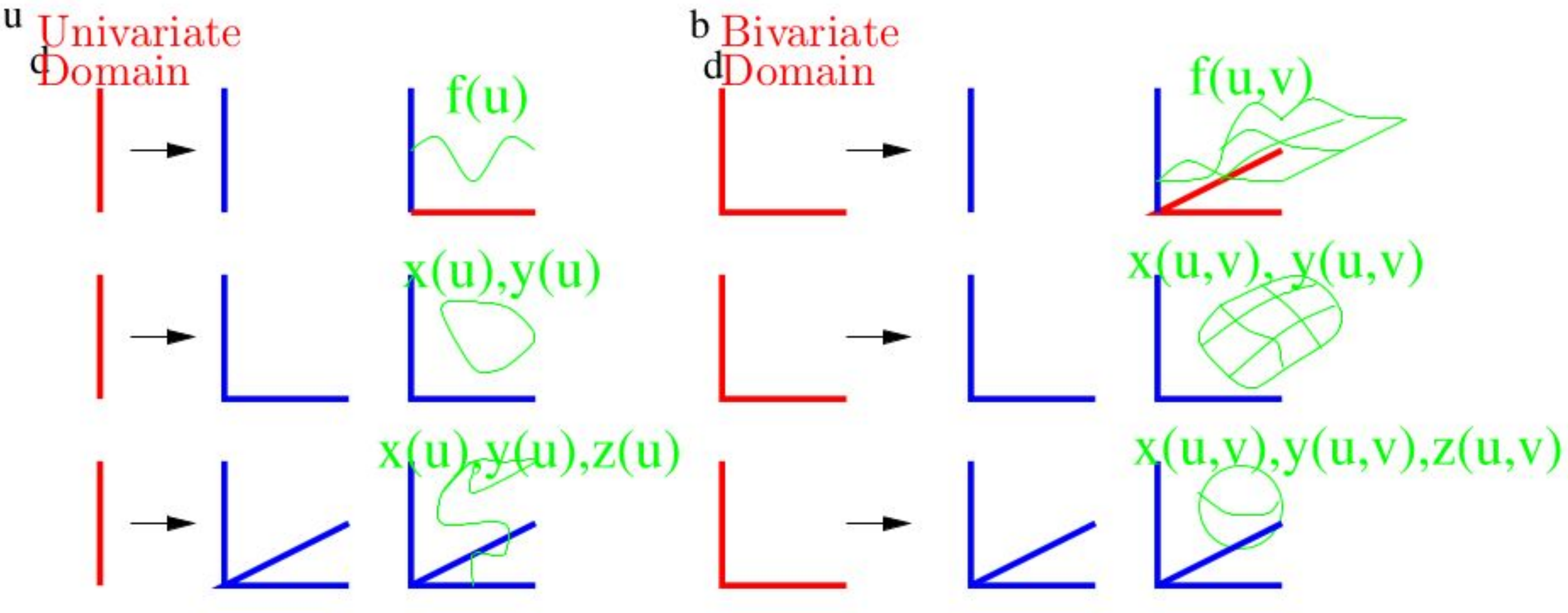
Maps and Curved Geometry

Domain, Range and Maps



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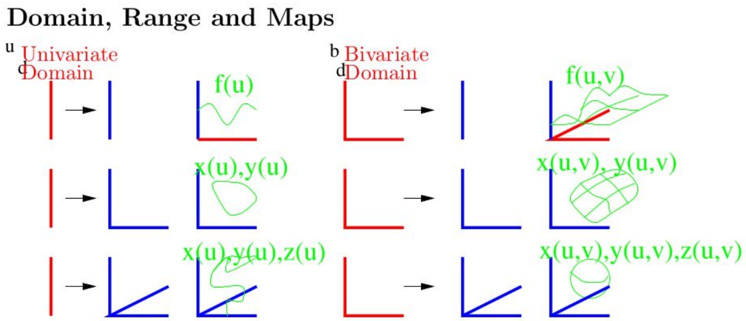

Related Concepts

- Graph of a function
- 1-Manifold ~ curve
- 2-Manifold ~ surface

- Scalar field: R^3 → R^1 Each point in 3-space is assigned a scalar
- Vector field: R^3 → R^3 ...a vector
- Tensor field: R^3 → R^3×3 ...a tensor

Maps and Curved Geometry

Polynomials and Polynomial Forms

Polynomial of degree n :-
an infinitely differentiable map whose nth derivative is constant

Polynomial p in of degree d in Bernstein-Bézier form **(BB-form)**

$$p := \sum_{i+j=d} c(i) B_{j,i}, \qquad \text{where } B_{j,i}(u) = \frac{d!}{i!j!} u^i (1-u)^j$$

Typically, p is evaluated on the interval [0, 1]. This yields a polynomial piece.

Curve editor https://www.desmos.com/calculator/ebdtbxgbq0

http://en.wikipedia.org/wiki/B%C3%A9zier_curve

BB-form

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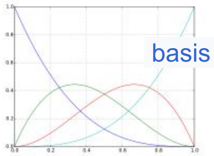

Maps and Curved Geometry

Computer Graphics Jorg Peters

Polynomials and Polynomial Forms

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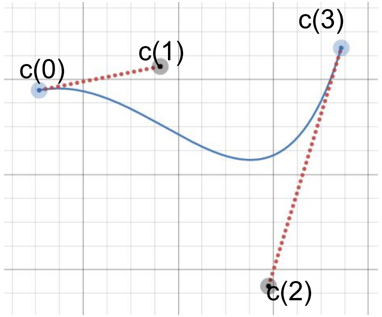
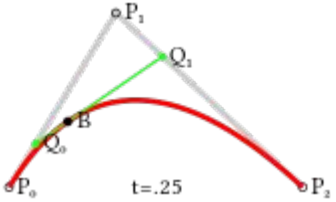
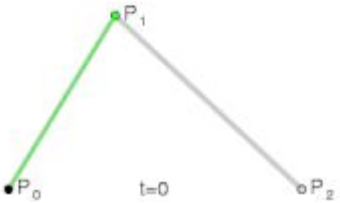


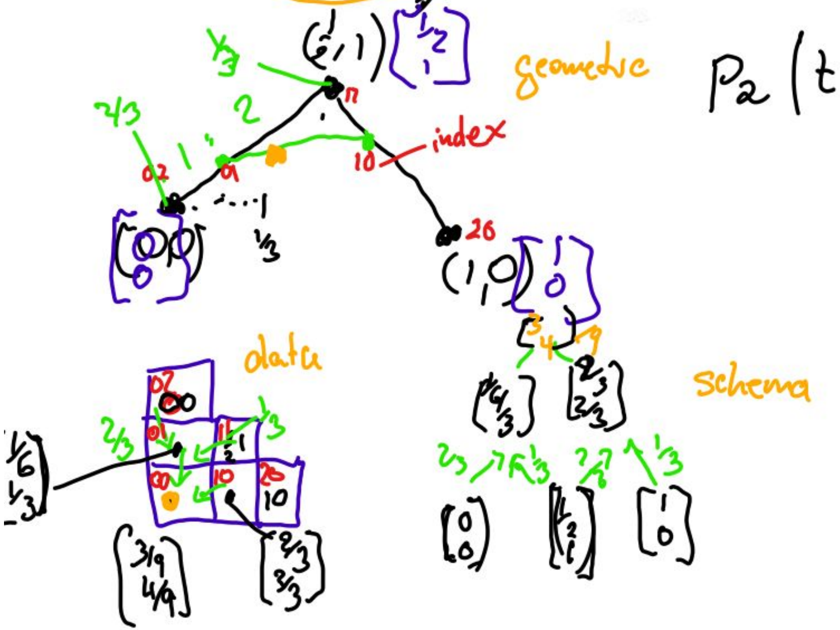




BB-form

$$p := \sum_{i+j=d} c(i) B_{j,i}, \qquad \text{where } B_{j,i}(u) = \frac{d!}{i!\,j!} u^i (1-u)^j$$

$$\binom{d}{i} = \frac{d!}{(d-i)!\,i!} = \frac{d!}{j!\,i!} = \binom{d}{j}$$



Algebraic

$$p_2\left(t=\tfrac13\right) = \sum_{i+j=2} c(i)\, B_{ij}\left(\tfrac13\right)$$

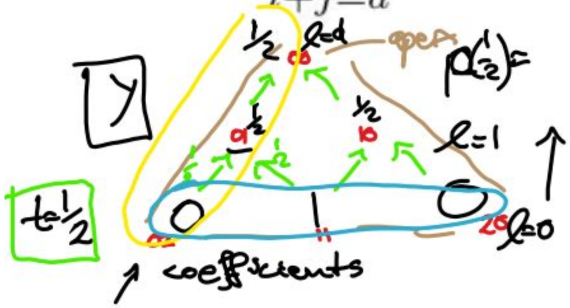
$$= \begin{bmatrix}0\\0\end{bmatrix}\frac{2!}{0!\,2!}\left(\tfrac13\right)^0\left(\tfrac23\right)^2 + \begin{bmatrix}\tfrac12\\1\end{bmatrix}\frac{2!}{1!\,1!}\left(\tfrac13\right)^1\left(\tfrac23\right)^1 + \begin{bmatrix}1\\0\end{bmatrix}\frac{2!}{2!\,0!}\left(\tfrac13\right)^2\left(\tfrac23\right)^0$$

$$= \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\tfrac12\\1\end{pmatrix} 2\,\frac29 + \begin{pmatrix}1\\0\end{pmatrix}\frac19 = \frac19\begin{pmatrix}2+1\\4\end{pmatrix} = \frac19\begin{pmatrix}3\\4\end{pmatrix}$$

$$= \begin{pmatrix}3/9\\4/9\end{pmatrix}$$

BB-form

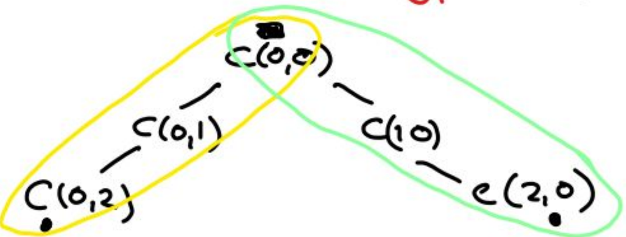
$$p := \sum_{i+j=d} c(i) B_{j,i}, \qquad \text{where } B_{j,i}(u) = \frac{d!}{i!j!} u^i (1-u)^j$$



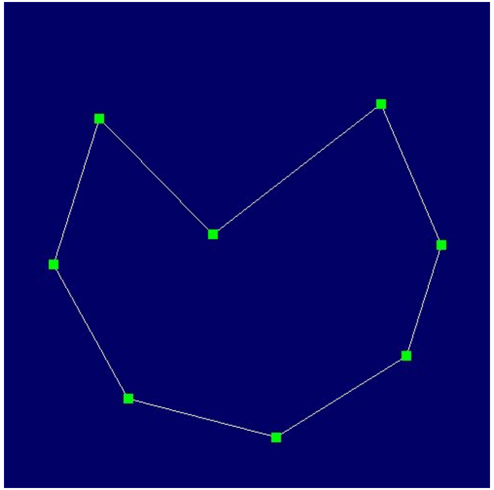
subdivision

algorithm de Casteljau

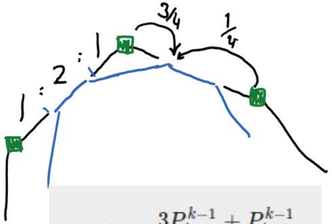
for l = 1..d
  for i = 0..d-l
    c(i, d-l-i) = (1-t) c(i, d-l-i+1) + t c(i+1, d-l-i)



B-form vs BB-form



Evaluate: De Boor's algorithm

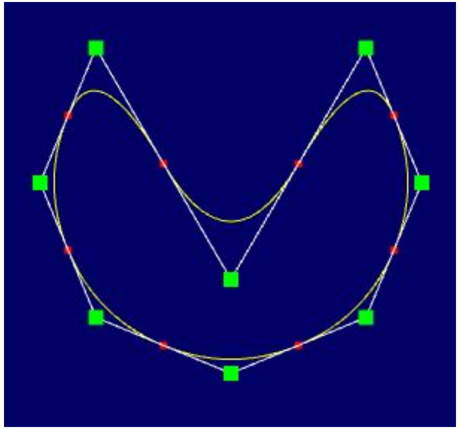


$$P^k_{2i} := \frac{3P^{k-1}_i + P^{k-1}_{i+1}}{4}$$

$$P^k_{2i+1} := \frac{P^{k-1}_i + 3P^{k-1}_{i+1}}{4}$$

Repeat = subdivision

≠



Evaluate: DeCasteljau's algorithm

Repeat = subdivision

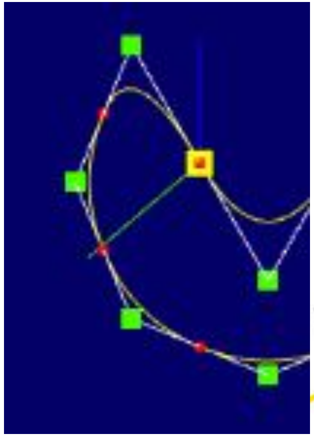
Maps and Curved Geometry

Piecewise polynomials in B-spline form (B-form)
Industry uses the acronym NURBS = Non-uniform rational B-spline

Uniform splines can be efficiently evaluated by subdivision.

Spline = piecewise polynomial (function).
Knots delineate the break points between polynomial pieces.
Typically some smoothness is enforced between the pieces.
A spline can be represented in Bézier form by connecting pieces in Bézier form.
A spline can be represented in B-spline form.

Space curves



$\mathbf{t} \sim \frac{d}{dt}\mathbf{x} \sim \mathbf{x}_1 - \mathbf{x}_0$

$\mathbf{m} = \mathbf{t} \times \mathbf{b}$

Project 1C:

osculating plane

$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$

$\mathbf{b} \sim (\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0)$