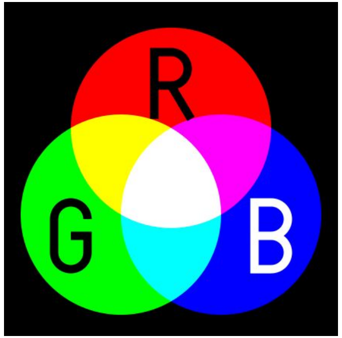
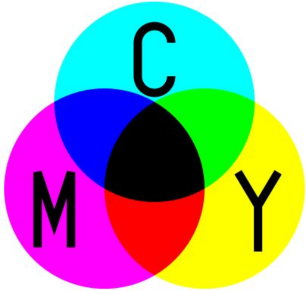
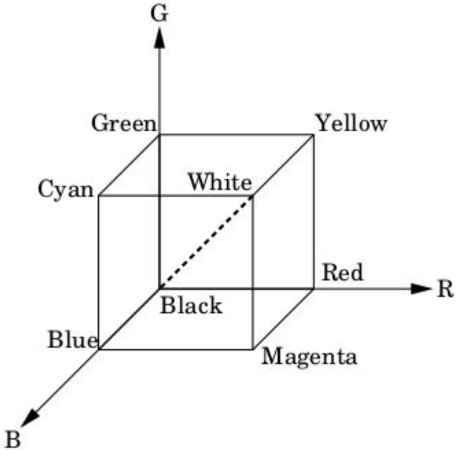
Coordinates: Color

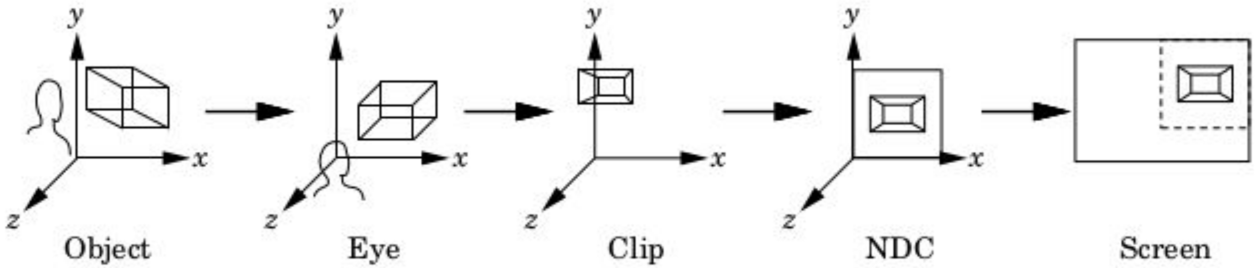






RGB vs Hue, Saturation, Intensity

Coordinates: OpenGL Pipeline



- ➢ model
- ➢ scene (world)
- ➢ eye (camera) [The Camera always looks down the negative z-axis.]
- ➢ clip (2-unit cube)
- ➢ normalized device (3D after perspective division)
- ➢ screen (after viewport transformation)

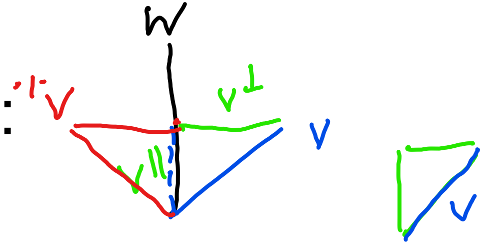
Euclidean Space: Rules

Inner product ($\cdot$)
Cross product ($\times$)

angles ($v \cdot w$) normalized
lengths ($v \cdot v$)
area ($v \times w$)
volume ($(u \times v) \cdot w$) $= \det(u,v,w)$

Euclidean Space

Working with Multiple Vectors



Projection of the vector v onto the vector w:

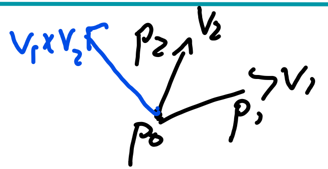
$p(v,w) := (v \cdot w \,/\, w \cdot w)\, w$

Perpendicular component of v to w:

$\tilde{v} := v - p(v,w) \quad \perp w$

Reflection of v across w: $v - 2\tilde{v}$

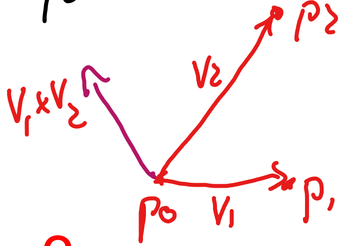
Euclidean Space Transformations



The linear map $M := [v_1, v_2, v_1 \times v_2]$, $\in \mathbb{R}^{3\times 3}$

$v_i := p_i - p_0$, $v_i := p_i - p_0$,

maps the

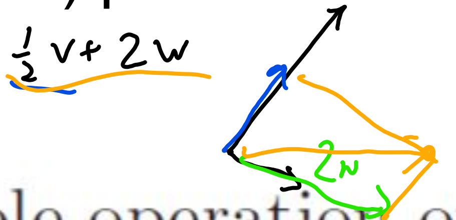


four points p0 ,p1 ,p2 ,p3 to four points p0 ,p1 ,p2 ,p3 :

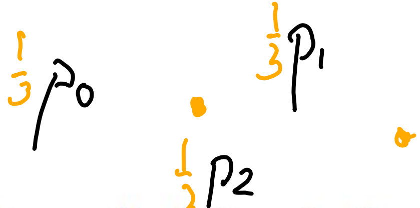
$$v3 = M M^{-1} v3, \quad \in R^{3\times 3}$$

Affine Coordinates

Elements: points (location) p and vectors (direction) v.



- $\sum_i \lambda_i \mathbf{v}_i$ is a vector;
  $\sum_i \lambda_i \mathbf{p}_i$ is an allowable operation only if $\sum_i \lambda_i = 1$.



  ▷ The Bernstein-Bézier form is well-defined since $\sum_i b_i(u) = 1$ for $b_i(u) := \binom{d}{i}(1-u)^{d-i}u^i$.

Affine Coordinates

Elements: points (location) p and vectors (direction) v.

- ▷ *Affine coordinates* in $\mathbb{R}^3$: append a 0 or 1 as 4th coordinate

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} := \begin{bmatrix} p_1 & p_2 & p_3 & 1 \end{bmatrix}^T ; \qquad \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} := \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & 0 \end{bmatrix}^T .$$

Affine Coordinates: Transformations

Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix} .$$

$$s := \sin(\theta), \; c := \cos(\theta)$$

Rotation

R_y: $\begin{pmatrix} c , & -s \\ s , & c \end{pmatrix}$

$$\begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} cx - sy \\ sx + cy \\ z \\ 1 \end{bmatrix}$$

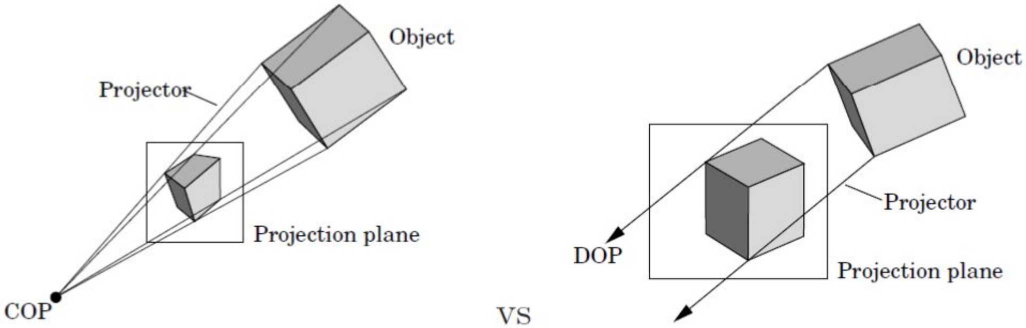
Rigid = translation, rotations, reflection

Planes and Quadrics

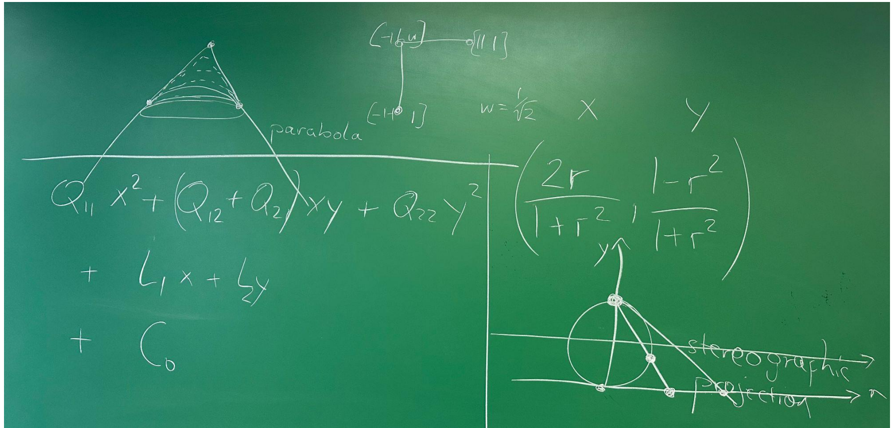
- Plane with normal $\mathbf{n}$, all (x, y, z) such that $\begin{bmatrix}\mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z & \mathbf{n}_c\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix} = 0$
- Conic, all (x, y) such that $\begin{bmatrix}x & y & 1\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12} & L_1\\Q_{12} & Q_{22} & L_2\\L_1 & L_2 & C_0\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix} = 0$
- Quadric, all (x, y, z) s.t. $\begin{bmatrix}x & y & z & 1\end{bmatrix}\begin{bmatrix}Q_{11} & Q_{12} & Q_{13} & L_1\\Q_{12} & Q_{22} & Q_{23} & L_2\\Q_{13} & Q_{23} & Q_{23} & L_3\\L_1 & L_2 & L_3 & C_0\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix} = 0$

Shortcomings of Affine Space

- While vectors can be added to vectors and points, points can only be added under extra constraints. Therefore there are two disjoint copies of Euclidean space.
- The rational Bernstein-Bézier form $\frac{\sum_i w_i \mathbf{p}_i b_i(u)}{\sum_i w_i b_i(u)} \simeq \sum_i [\begin{smallmatrix} w_i \mathbf{p}_i \\ w_i \end{smallmatrix}] b_i(u)$ cannot be represented because generally $w_i \notin \{0,1\}$.
- Perspective projection cannot be represented

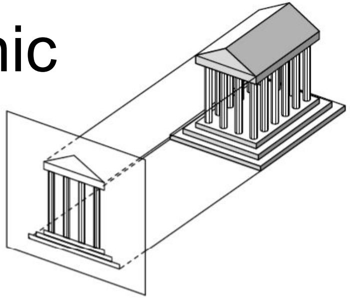


Circle



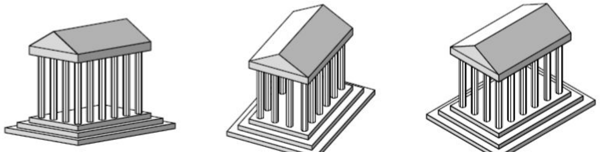
Projections

Orthographic Projection



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(dimetric: two of the three axes of space appear equally foreshortened)



Dimetric Trimetric Isometric

3-,2-, and 1-point perspective (=number of vanishing points).
http://www.termespheres.com/perspective.html

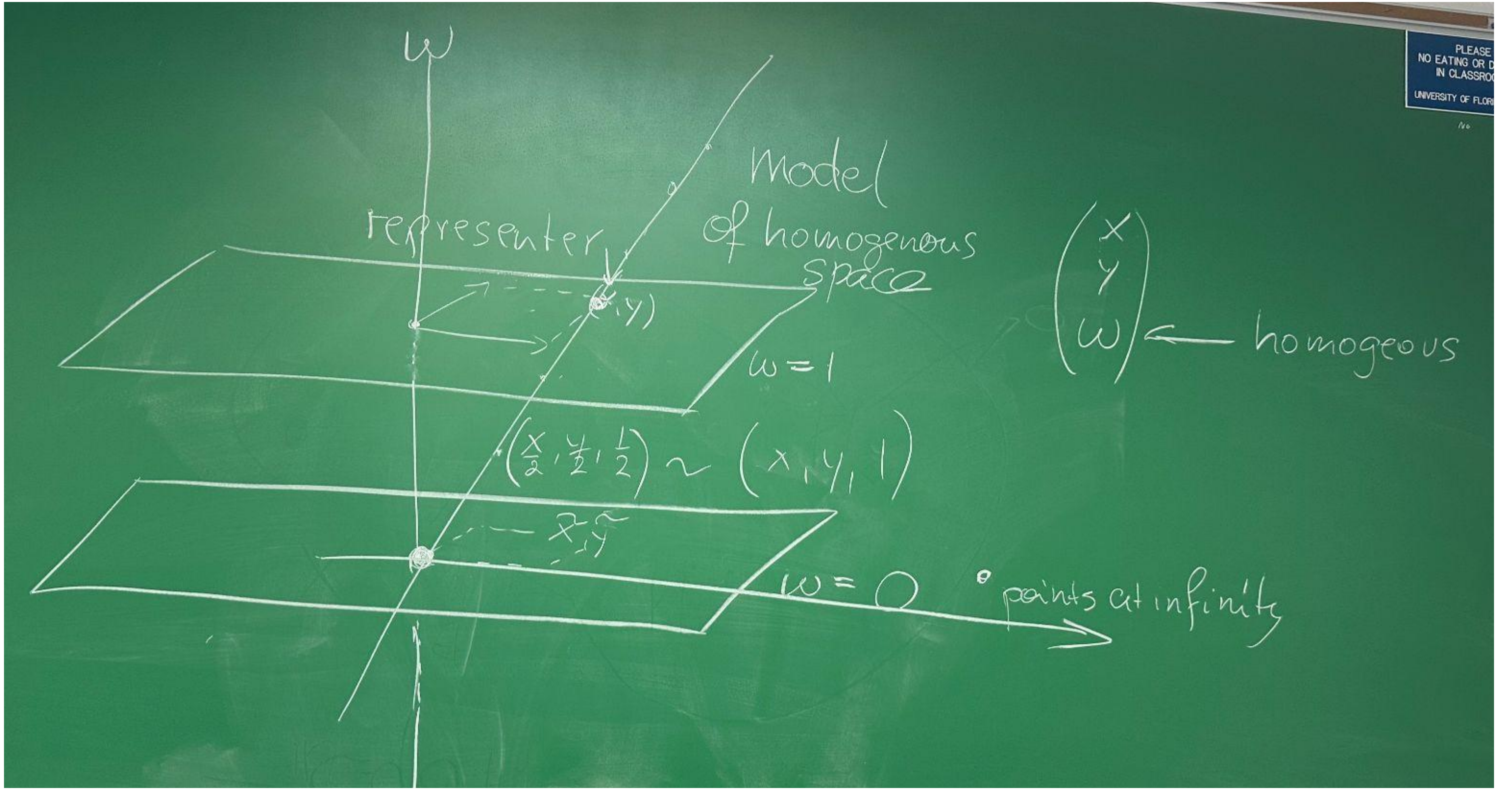
(a) (b) (c)

Projective Space, Homogeneous Coord's

- ▷ an *affine point* $\begin{bmatrix} x_1 & x_2 & \dots & x_n & x_{n+1} \end{bmatrix}^T$ if $x_{n+1} \neq 0$;
- ▷ a *point at infinity* when $\begin{bmatrix} x_1 & x_2 & \dots & x_n & 0 \end{bmatrix}^T$.

- The homogeneous representation 'completes the geometry': A pair of lines always intersects in a point, possibly at infinity.
  (advanced) *More generally,* Bezout's Theorem *holds in complex projective space: for polynomials* $p(x, y)$ *and* $q(x, y)$ *that have no common factor and are of degree* m *and* n *repectively, the curves* $p(x, y) = 0$ *and* $q(x, y) = 0$ *have* mn *intersections.*

Projective Space, Homogeneous Coord's

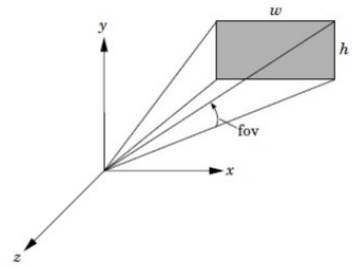


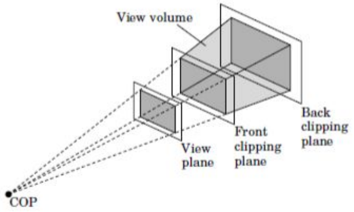
Projective Space: Perspective

- *Perspective Scaling*

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1/k\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix} = \begin{bmatrix}x\\ y\\ z\\ 1/k\end{bmatrix} = \begin{bmatrix}kx\\ ky\\ kz\\ 1\end{bmatrix}.$$

- *glPerspective*(fovy,aspect,near,far)
  Modelview → Projection → Perspective Division





Projective Space: Frustum

▷ *Frustum, Clipping* (Projection into 3D double unit box) with $\Delta x := \bar{x} - \underline{x} =$ (xmax-xmin) etc.

$$glFrustum(\text{xmin,xmax,ymin,ymax,near,far}) \quad \begin{bmatrix} \frac{2}{\Delta x} & 0 & 0 & -\frac{\bar{x}+\underline{x}}{\Delta x} \\ 0 & \frac{2}{\Delta y} & 0 & -\frac{\bar{y}+\underline{y}}{\Delta y} \\ 0 & 0 & \frac{2}{\Delta z} & -\frac{\bar{z}+\underline{z}}{\Delta z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Maps a viewing frustum (rectangular box) to $[-1,1]^3$.

Example: (A point between $\underline{x}$ and $\bar{x}$ is mapped to the interval $[-1,1]$)

$\underline{x} = -4,\ \bar{x} = 0, \qquad \Delta x = 4, \qquad -\frac{\bar{x}+\underline{x}}{\Delta x} = 1, \qquad \frac{2}{\Delta x} = 1/2.$

Then $\begin{bmatrix} -3 \\ * \\ * \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} -1/2 \\ * \\ * \\ 1 \end{bmatrix}$.

Projective Space: not a vector space

- A point on a projective line does not split the line into two parts: a person looking in one unobstructed direction sees his on back.
  Construction of curves is unintutive.

- There is no notion of length.

- It is not a vector space. Addition of homogeneous coordinates (here $n = 2$) does not work:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix}, \quad \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}9\\6\\3\end{bmatrix}, \quad \text{but} \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}4\\4\\4\end{bmatrix} \neq \begin{bmatrix}11\\10\\9\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} + \begin{bmatrix}9\\6\\3\end{bmatrix}.$$