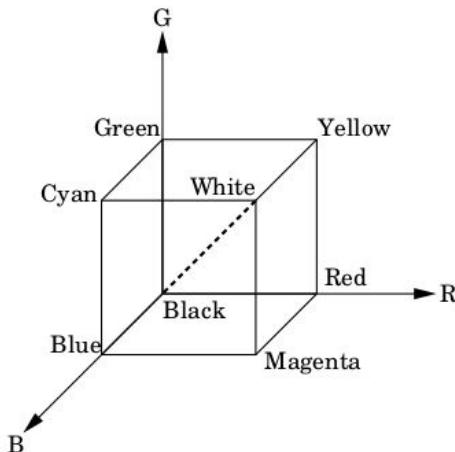
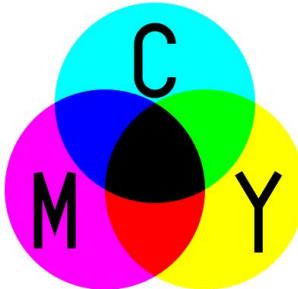
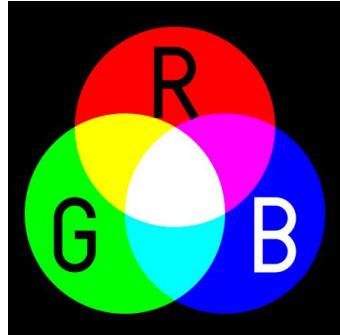
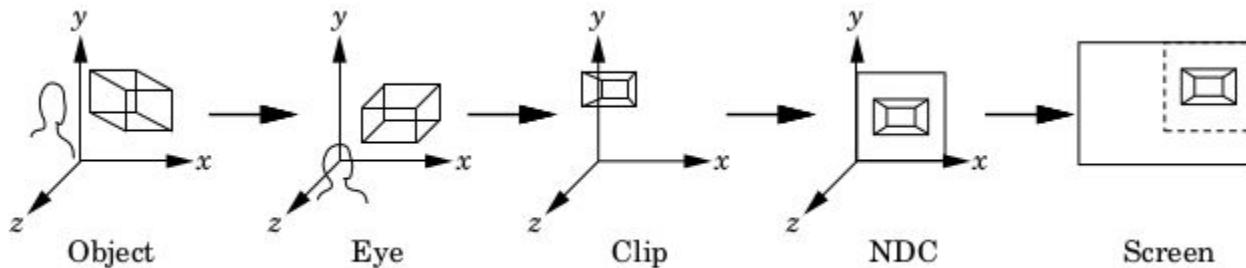


Coordinates: Color



RGB vs Hue, Saturation, Intensity

Coordinates: OpenGL Pipeline



- model
- scene (world)
- eye (camera) [The Camera always looks down the negative z-axis.]
- clip (2-unit cube)
- normalized device (3D after perspective division)
- screen (after viewport transformation)

Euclidean Space: Rules

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Inner product (\cdot)

Cross product (\times)

angles ($v \cdot w$) normalized

lengths ($v \cdot v$)

area ($v \times w$) For v, w in \mathbb{R}^2

volume ($(u \times v) \cdot w$) = $\det(u, v, w)$ For u, v, w in \mathbb{R}^3

Euclidean Space

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Working with Multiple Vectors

Projection of the vector v onto the vector w :

$$p(v,w) := (v \cdot w / w \cdot w) w$$

Perpendicular component of v to w :

$$\tilde{v} := v - p(v,w) \perp w$$

Reflection of v across w : $v - 2\tilde{v}$

Euclidean Space Transformations

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The linear map $M := [v_1, v_2, v_1 \times v_2]$,
 $v_i := p_i - p_0$, $v_i := p_i - p_0$,
maps the

four points p_0, p_1, p_2, p_3 to four points p_0, p_1, p_2, p_3 :
 $v_3 = M M^{-1} v_3, \in \mathbb{R}^{3 \times 3}$

Affine Coordinates

Elements: points (location) p and vectors (direction) v .

- $\sum_i \lambda_i \mathbf{v}_i$ is a vector;
 $\sum_i \lambda_i \mathbf{p}_i$ is an allowable operation only if $\sum_i \lambda_i = 1$.
- ▷ The Bernstein-Bézier form is well-defined since $\sum_i b_i(u) = 1$ for $b_i(u) := \binom{d}{i} (1-u)^{d-i} u^i$.

Affine Coordinates

Elements: points (location) \mathbf{p} and vectors (direction) \mathbf{v} .

- ▷ *Affine coordinates* in \mathbb{R}^3 : append a 0 or 1 as 4th coordinate

$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} := [p_1 \quad p_2 \quad p_3 \quad 1]^T ; \quad \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} := [\tilde{v}_1 \quad \tilde{v}_2 \quad \tilde{v}_3 \quad 0]^T .$$

Affine Coordinates: Transformations

Translation

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}.$$

$$s := \sin(\theta), c := \cos(\theta)$$

Rotation

$$\begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} cx - sy \\ sx + cy \\ z \\ 1 \end{bmatrix}$$

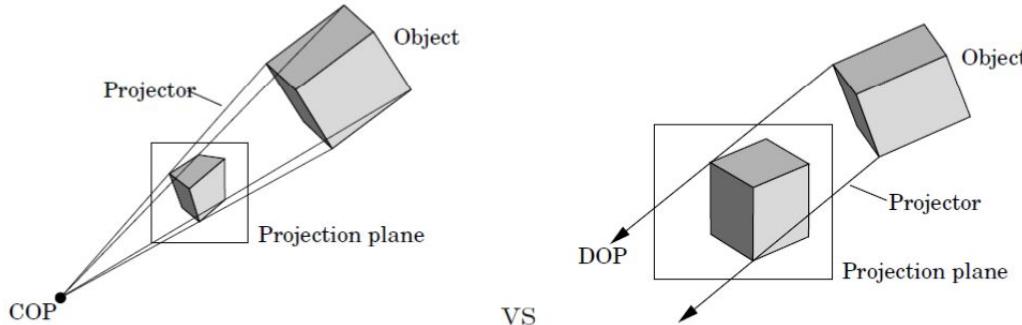
Rigid = translation, rotations, reflection

Planes and Quadrics

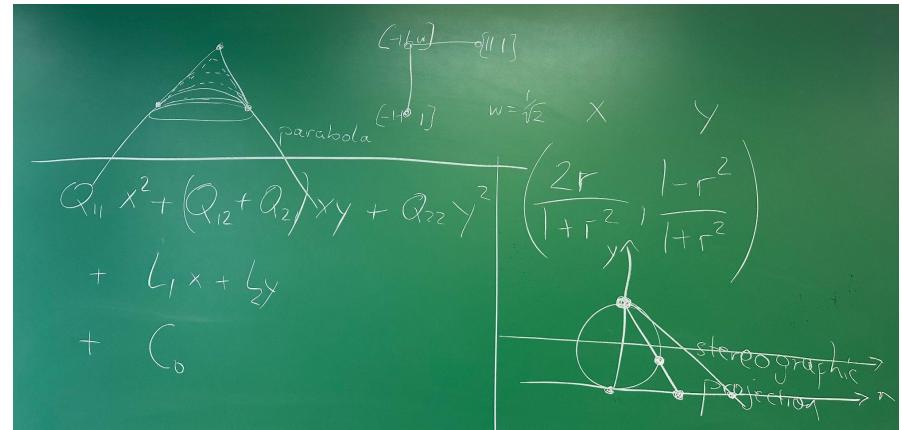
- ▷ Plane with normal \mathbf{n} , all (x, y, z) such that $[\mathbf{n}_x \quad \mathbf{n}_y \quad \mathbf{n}_z \quad \mathbf{n}_c] \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$
- ▷ Conic, all (x, y) such that $[x \quad y \quad 1] \begin{bmatrix} Q_{11} & Q_{12} & L_1 \\ Q_{12} & Q_{22} & L_2 \\ L_1 & L_2 & C_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$
- ▷ Quadric, all (x, y, z) s.t. $[x \quad y \quad z \quad 1] \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & L_1 \\ Q_{12} & Q_{22} & Q_{23} & L_2 \\ Q_{13} & Q_{23} & Q_{33} & L_3 \\ L_1 & L_2 & L_3 & C_0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$

Shortcomings of Affine Space

- While vectors can be added to vectors and points, points can only be added under extra constraints. Therefore there are two disjoint copies of Euclidean space.
- The rational Bernstein-Bézier form $\frac{\sum_i w_i \mathbf{p}_i b_i(u)}{\sum_i w_i b_i(u)} \simeq \sum_i [{}^w_i \mathbf{p}_i] b_i(u)$ cannot be represented because generally $w_i \notin \{0, 1\}$.
- Perspective projection cannot be represented



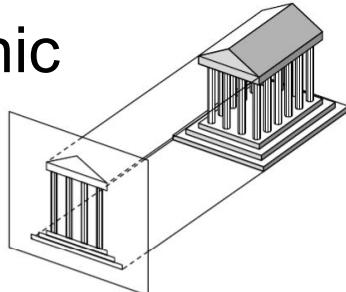
Circle



Projections

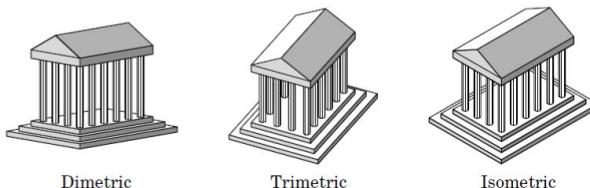
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Orthographic Projection

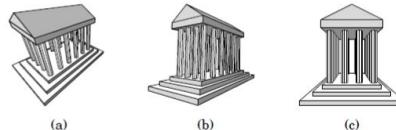


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(dimetric: two of the three axes of space appear equally foreshortened)



3-, 2-, and 1-point perspective (=number of vanishing points).
<http://www.termespheres.com/perspective.html>

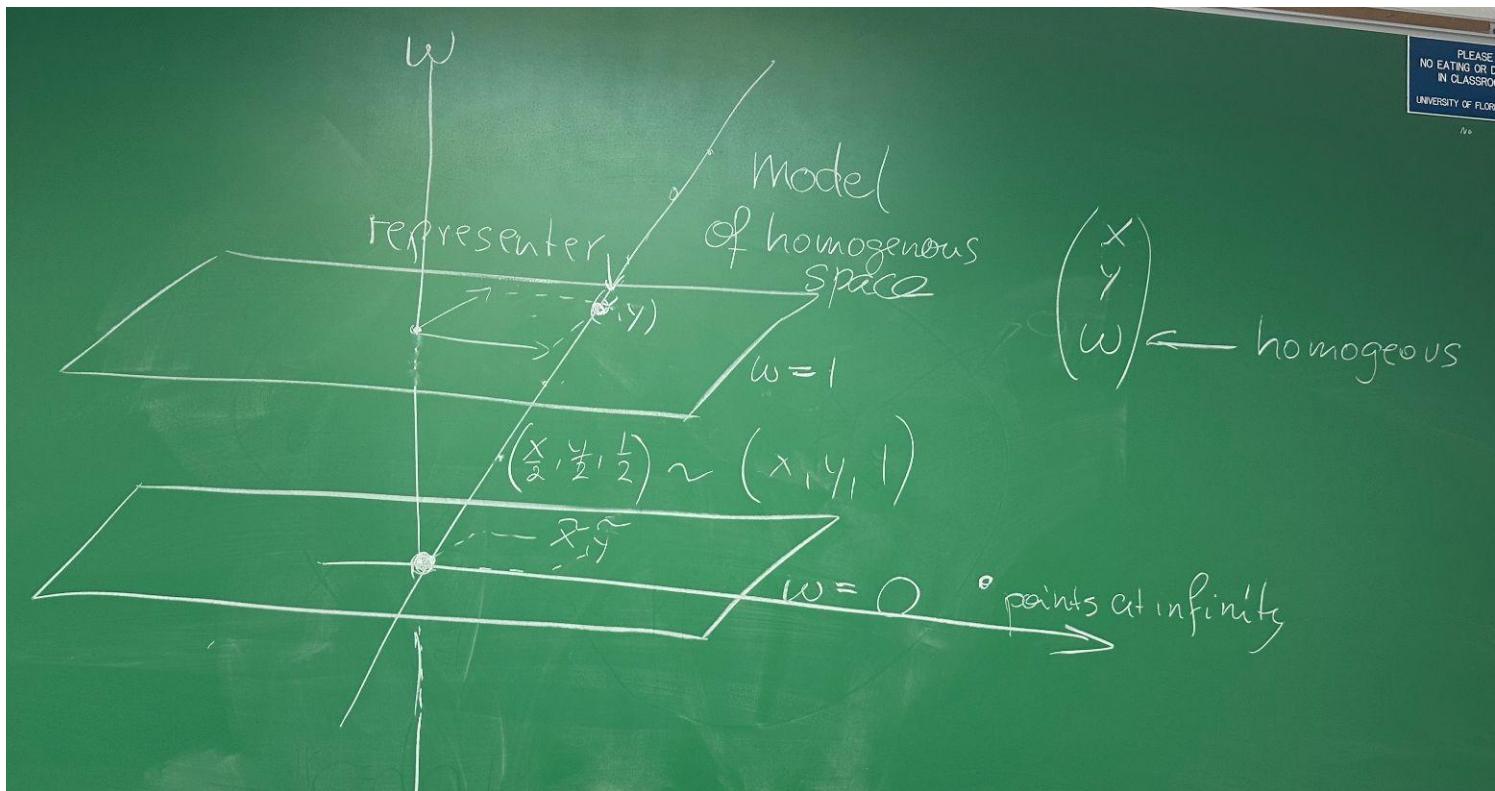


Projective Space, Homogeneous Coord's

- ▷ an *affine point* $[x_1 \ x_2 \ \dots \ x_n \ x_{n+1}]^T$ if $x_{n+1} \neq 0$;
- ▷ a *point at infinity* when $[x_1 \ x_2 \ \dots \ x_n \ 0]^T$.
- The homogeneous representation ‘completes the geometry’: A pair of lines always intersects in a point, possibly at infinity.
(advanced) *More generally, Bezout’s Theorem holds in complex projective space: for polynomials $p(x, y)$ and $q(x, y)$ that have no common factor and are of degree m and n respectively, the curves $p(x, y) = 0$ and $q(x, y) = 0$ have mn intersections.*

Projective Space, Homogeneous Coord's

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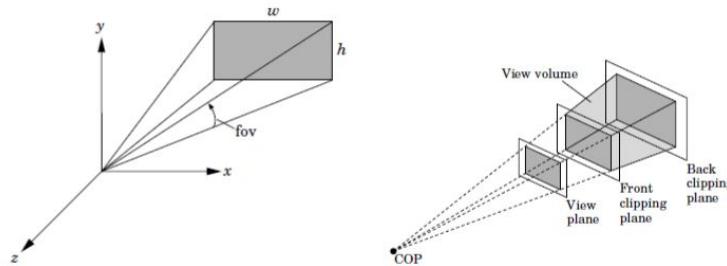


Projective Space: Perspective

▷ *Perspective Scaling*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1/k \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kz \\ 1 \end{bmatrix}.$$

▷ `glPerspective(fovy, aspect, near, far)`
Modelview → Projection → Perspective Division



Projective Space: Frustum

- ▷ *Frustum, Clipping* (Projection into 3D double unit box) with $\Delta x := \bar{x} - \underline{x} = (\text{xmax}-\text{xmin})$ etc.

`glFrustum(xmin,xmax,ymin,ymax,near,far)`

$$\begin{bmatrix} \frac{2}{\Delta x} & 0 & 0 & -\frac{\bar{x}+\underline{x}}{\Delta x} \\ 0 & \frac{2}{\Delta y} & 0 & -\frac{\bar{y}+\underline{y}}{\Delta y} \\ 0 & 0 & \frac{2}{\Delta z} & -\frac{\bar{z}+\underline{z}}{\Delta z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Maps a viewing frustum (rectangular box) to $[-1, 1]^3$.

Example: (A point between \underline{x} and \bar{x} is mapped to the interval $[-1, 1]$)

$$\underline{x} = -4, \bar{x} = 0, \quad \Delta x = 4, \quad -\frac{\bar{x}+\underline{x}}{\Delta x} = 1, \quad \frac{2}{\Delta x} = 1/2.$$

Then $\begin{bmatrix} -3 \\ * \\ * \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} -1/2 \\ * \\ * \\ 1 \end{bmatrix}$.

Projective Space: not a vector space

- A point on a projective line does not split the line into two parts: a person looking in one unobstructed direction sees his own back.
Construction of curves is unintuitive.
- There is no notion of length.
- It is not a vector space. Addition of homogeneous coordinates (here $n = 2$) does not work:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \quad \text{but} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 11 \\ 10 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}.$$