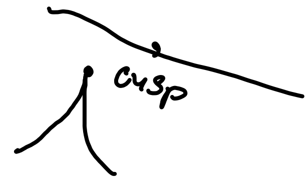
$\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix} = \begin{bmatrix}x'\\ y'\end{bmatrix}$ gradient



The univariate Bernstein-Bézier form

a Definition

The Bernstein basis functions of degree n on the interval $[0, 1]$ are

$$b_{n-i,i} : \mathbb{R} \to \mathbb{R}, u \mapsto \binom{n}{i}(1-u)^{n-i}u^i.$$

Note 1: When the degree n is understood, we drop the subscript $n-i$.
Note 2: Sometime we use t as the parameter (to remind of 'time'), sometimes u (in anticipation of using two parameters). It makes no difference.
A polynomial of degree n in Bernstein-Bézier form (BB-form) has the representation

$$\mathbf{p}_n(u) := \sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}(u) \qquad \sim_{\text{short}} \mathbf{p}_n(u) := \sum_{i=0}^{n} \mathbf{c}_i b_i(u). \tag{.1}$$

The *BB-coefficients* $\mathbf{c}_{n-i,i} \in \mathbb{R}^d$ can have d coordinates and are often called control points, because of their nice geometric properties (see below). We reserve the term control point for another, equivalent representation, called B-spline representation. [1] We are interested in the polynomial piece traced out when

$$u \in [0..1].$$

(Otherwise see *reparametrization* below).

b Symmetry

The symmetry with respect to $u = 1/2$ is nicely captured by introducing the abbreviation $v := 1-u$. This yields the equivalent representation

$$b_{j,i}(u, v) := \frac{n!}{i!j!}u^i v^j, \qquad \text{where } u + v = 1, i + j = n,$$

with the obvious symmetry $b_{n-i,i}(u) = b_{i,n-i}(1-u)$.

[1] If $\mathbf{c}_{n-i,i} = f(i/n)$ for some continuous function f then the polynomial is called *Bernstein polynomial of f*. This special type of polynomial in Bernstein form popular in analysis will only be a footnote.

c Partition of unity and positivity

You will check below that the BB-basis functions add to 1 and are nonnegative:

$$\sum_{i=0}^{n} b_{n-i,i} = 1, \tag{.2}$$

$$1 \geq b_{n-i,i}(u) \geq 0 \text{ for } u \in [0..1]. \tag{.3}$$

If you fix $u \in [0..1]$, say $u = 1/2$, each $b_{n-i,i}(u)$ is a number between 0 and 1 and (.1) tells us to form a convex average of the control points $\mathbf{c}_{n-i,i}$. Therefore the polynomial (curve) is in the *convex hull* of the coefficients when $u \in [0, 1]$.

(.3) also implies that the BB-form is *affinely invariant*. To see what this means let $A \in \mathbb{R}^{d\times d}$ be a $d \times d$ matrix (for example a 2×2 rotation matrix) and $\mathbf{a} \in \mathbb{R}^d$ be a vector with d coordinates. Then

$$\mathbf{a} + A\sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}(u) = \sum_{i=0}^{n} (\mathbf{a} + A\mathbf{c}_{n-i,i}) b_{n-i,i}(u). \tag{.4}$$

That is, we obtain the same result whether we transform (by applying A and translating by $\mathbf{a}$) the curve $\sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}(u)$ or whether we transform the control points $\mathbf{c}_{n-i,i}$ and then form the curve.

..1– Exercise [3]: Prove that $\sum_{i=0}^{n} b_{n-i,i}(u) = 1$, and $1 \geq b_{n-i,i}(u) \geq 0$.

..2– Exercise [5]: Prove that $b_{n-i,i}$ has its maximum over $[0 \,..\, 1]$ at the *Greville abscissa* $u_i := i/n$. (Hint: where is the derivative zero?)

..3– Exercise [5]: Draw the basis functions when $n = 3$ $b_{n-i,i}$ for $i = 0..3$ on the interval $[0..1]$.

d Evaluation

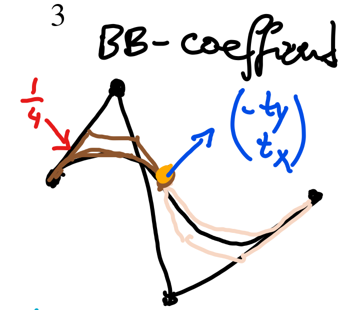
Evaluation = iterated linear interpolation

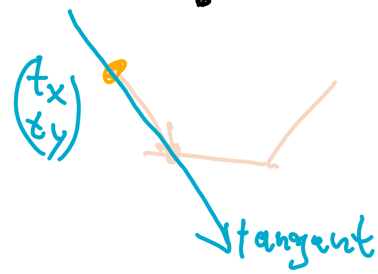
The corresponding *nested multiplication* for evaluating the polynomial

$$\mathbf{p}_n(u) := \sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}(u) \tag{.5}$$

at $u = x$ is called
De Casteljau's algorithm:

for $i = 0 : n, p(n-i, i) \leftarrow \mathbf{c}_{n-i,i}$ ...fill array
for $l = 1 : n,$ ...for each level
for $i = 0 : n - l,$ ...from left to right
$p(n-l-i, i) = (1-x)\, p(n-l+1-i, i)$
$+ x\, p(n-l-i, i+1)$
end
end
$p_n(x) \leftarrow p(0, 0).$





e Subdivision

Subdivision = evaluation pyramid roof

De Casteljau's algorithm does more than generate the value. The identity

$$b_{n-i,i}(xu) = \sum_{k \geq i}^{n} b_{k-i,i}(x) b_{n-k,k}(u)$$

implies that the restriction of $\mathbf{p}_n$ to the interval $[0, x]$ is given by

$$\begin{aligned}
\mathbf{p}_{[0,x]}(u) &:= \sum_{i=0}^{n} a(i) b_{n-i,i}(xu) \\
&= \sum_{i=0}^{n} \left(\sum_{k=i}^{n} a(i) b_{k-i,i}(x) \right) b_{n-k,k}(u) \\
&= \sum_{k=0}^{n} \left(\sum_{i=0}^{k} a(i) b_{k-i,i}(x) \right) b_{n-k,k}(u) \\
&= \sum_{k=0}^{n} p(\boxed{k, 0}) b_{n-k,k}(u).
\end{aligned}$$

f Differentiation

Differentiation = Differencing coefficients

The derivative $\partial \mathbf{p}_n$ of a polynomial $\mathbf{p}_n$ in Bernstein form must be writable as a polynomial of one degree less in Bernstein form:

$$\partial\Big(\sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}\Big) = \sum_{i=0}^{n-1} \mathbf{d}_{n-1-i,i} b_{n-1-i,i}$$

..4– Exercise [5]: Show that the coefficients of the derivative are

$$\mathbf{d}_{n-i,i} := n(\mathbf{c}_{n-i-1,i+1} - \mathbf{c}_{n-i,i}).$$

..5– Exercise [5]: Show that under the change variables $u \to \frac{x-a}{b-a}$, the coefficients become

$$\mathbf{d}_{n-i,i} := \frac{n}{b-a}(\mathbf{c}_{n-i-1,i+1} - \mathbf{c}_{n-i,i}).$$

..6– Example: The second derivative of the quadratic polynomial

$$\mathbf{p}_2 = 3b_{0,2} + 5b_{1,1} + 8b_{2,0}$$

is

$$\begin{aligned}
\mathbf{p}_2(u) &= 3(1-u)^2 + 5 * 2(1-u)u + 8u^2 \\
\partial\mathbf{p}_2(u) &= 2(5-3)b_{0,1} + 2(8-5)b_{1,0} \\
&= 4(1-u) + 6u \\
\partial^2\mathbf{p}_2(u) &= 1 * (6-4)b_{0,0} \\
&= 2.
\end{aligned}$$

Indeed, applying the generic rules of differentiation to $\mathbf{p}_2$, we get

$$\partial^2\mathbf{p}_2 = 3 * 2 + 5 * (-4) + 8 * 2 = 2.$$

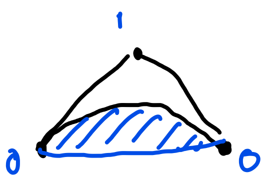
..7– Exercise [5]: Give a formula for coefficients $\mathbf{e}_{n-i,i}$ such that

$$\partial^2\Big(\sum_{i=0}^{n} \mathbf{c}_{n-i,i} b_{n-i,i}\Big) = \sum_{j=0}^{n-2} \mathbf{e}_{n-2-j,j} b_{n-2-j,j}.$$

g Integration

Integration = Summing coefficients

..8– Exercise [10]: Show that



$$\int_0^1 \sum \mathbf{c}_{n-i,i} b_{n-i,i} du = \sum \mathbf{c}_{n-i,i}/(n+1).$$

$(0+1+0)/3$