Approximation Algorithms And Hardness For Domination With Propagation

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Outlines

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2. Problem Definitions
   - Power Dominating Set (PDS)
   - ℓ-round PDS

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4. $\sqrt{n}$-Approximation Algorithm of PDS on Planar Graphs

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5. PTAS for ℓ-round PDS on Planar Graphs
6. PDS in Directed Graphs

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- In order to monitor a power network we need to measure all the state variables of the network by placing measurement devices
- A Phasor Measurement Unit (PMU) is a measurement device placed on a node that has the ability to measure the voltage of the node and the current phase of the edges connected to the node
- These units have the capability of monitoring remote elements via propagation (Rule 2)

Power Dominating Set (PDS)

Power Dominating Set

The Power Dominating Set problem (PDS) is a covering problem in which the goal is to power dominate (cover) all the nodes of a given undirected graph G by picking as few nodes as possible. Given a set of nodes S, the set of nodes that are power dominated by S, denoted $P(S)$. There are two rules as follows:

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'Local' Effect
: if node v is in S, then v and all of its neighbors are in $P(S)$;

Propagation
: if node v is in $P(S)$, one of its neighbors w is not in P(S), and all other neighbors of v are in $P(S)$, then w is inserted into $P(S)$.

ℓ-round PDS

Parallel Propagation Rule

Given a graph $G = (V, E)$ and a subset of nodes $S \subseteq V$, the set of nodes that can be power dominated by applying at most k rounds of parallel propagation, denoted by $P^k(S)$, is defined recursively as follows:

$$P^k(S) = \begin{cases} \cup_{v \in S} N[v], & k = 1 \\ P^{k-1}(S) \cup \{v : (u, v) \in E, N[u] \setminus \{v\} \subseteq P^{k-1}(S), & k \geq 2 \end{cases}$$

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ℓ-round PDS

Given a parameter ℓ, the ℓ-round PDS problem is the problem in which we are given a graph $G = (V, E)$ and the goal is to find a minimum size subset of nodes $S \subseteq V$, such that $P^\ell(S) = V$.

Example

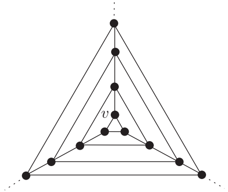

Fig. 1. Example for PDS

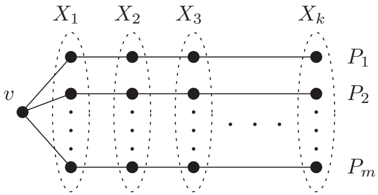

Fig. 2. Example for ℓ-round PDS

Approximation Hardness of PDS and ℓ-round PDS

Theorem

The PDS (and ℓ-round PDS for $\ell \geq 4$) problem cannot be approximated within ratio $2^{\log^{1-\epsilon} n}$, for any $\epsilon > 0$, unless $NP \subseteq DTIME\left(n^{poly \log n}\right)$.

Proof.

The idea is to show the gap-preserving from MinRep Problem to PDS. □

Detail Proof

MinRep Problem:

In the MinRep problem we are given a bipartite graph $G = (A, B, E)$ with a partitioning of A and B into (equal size) subsets, say $A = \bigcup_{i=1}^{q_A} A_i$ and $B = \bigcup_{i=1}^{q_B} B_i$, where $|A_i| = m_A = \frac{|A|}{q_A}$ and $|B_i| = m_B = \frac{|B|}{q_B}$. This partitioning naturally defines a super bipartite graph $\mathcal{H} = (\mathcal{A}, \mathcal{B}, \mathcal{E})$. The super nodes of H are $\mathcal{A} = \{A_1, A_2, \cdots, A_{q_A}\}$ and $\mathcal{B} = \{B_1, B_2, \cdots, B_{q_B}\}$, and the super edges are $\mathcal{E} = \{A_iB_j | \exists a \in A_i, b \in B_j : ab \in E(G)\}$. We say that super edge A_iB_j is covered by $ab \in E(G)$ if $a \in A_i$ and $b \in B_j$. The goal in MinRep is to pick the minimum number of nodes, $A' \cup B' \subseteq V(G)$, from G such that all the super edges in $\mathcal{H}$ are covered. The following theorem states the hardness of the MinRep problem [20].

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Theorem

MinRep cannot be approximated within ratio $2^{\log^{1-\epsilon} n}$, for any $\epsilon > 0$, unless $NP \subseteq DTIME\left(n^{poly \log n}\right)$, where $n = |V(G)|$.

Detail Proof (Cont.)

The Reduction:

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The Reduction:

1. Add a new node w^* (master node) to the graph G, and add an edge between w^* and all the nodes in G. Also add new nodes w_1^*, w_2^*, w_3^* and connect them by an edge to w^*.
2. $\forall i \in \{1, \dots, q_A\}, j \in \{1, \dots, q_B\}$ do the following:
   (a) Let $E_{ij}=\{e_1, e_2, \dots, e_\kappa\}$ be the set of edges between $A_i = \left\{a_{i_1}, \dots, a_{i_{m_A}}\right\}$ and $B_j = \left\{b_{j_1}, \dots, b_{j_{m_B}}\right\}$ in G, where κ is the number of edges between A_i and B_j.
   (b) Remove E_{ij} from G.
   (c) Let the edge $e_q \in E_{i,j}$ be incident to a_{i_q} and b_{j_q} (in G). In this labeling for simplicity the same node might get different labels. Let D_{ij} be the graph in Figure 3 (A dashed line shows an edge between the master node w^* and that node). Make $k = 4$ new copies of the graph D_{ij} and then identify nodes a_{i_q}'s, b_{j_q}'s with the corresponding nodes in A_i and B_j (in G). Note that the k copies are sharing the same set of nodes, A_i and B_j, but other nodes are disjoint.
3. Let $\overline{G} = (\overline{V}, \overline{E})$ be the obtained graph.

Detail Proof (Cont.)

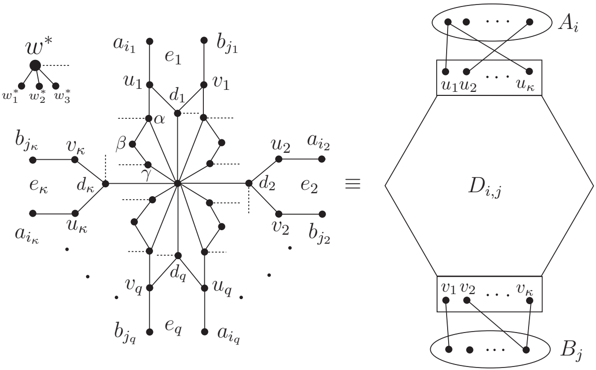

Fig. 3. D_{ij} graph

Detail Proof (Cont.)

Lemma

(A^*, B^*) is an optimal solution to the instance $G = (A, B, E)$ of the MinRep problem if and only if $\Pi^* = A^* \cup B^* \cup \{w^*\} \subseteq V(\bar{G})$ is an optimal solution to the instance $\bar{G}$ of PDS (and ℓ-round PDS for all $\ell \geq 4$).

Proof.

The proof follows in two ways. $\square$

$\sqrt{n}$-Approximation Algorithm of PDS on Planar Graphs

Idea

- The idea is to use the tree decomposition since the tree-width of a planar graph G with n nodes is $O(\sqrt{n})$ which can be found in $O(n^{3/2})$ time.

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Idea

- The idea is to use the tree decomposition since the tree-width of a planar graph G with n nodes is $O(\sqrt{n})$ which can be found in $O(n^{3/2})$ time.
- The key point is to show that $Opt(G) \geq m$, where m is the number of nodes of T where we updated Π.

Some Definitions

Definition 1. *A tree decomposition of a graph* $G = (V, E)$ *is a pair* $\langle \{X_i \subseteq V \mid i \in I\}, T\rangle$, *where* $T = (I, F)$ *is a tree, satisfying the following three properties: (1)* $\bigcup_{i \in I} X_i = V$; *(2) For all edges* $\{u, v\} \in E$ *there is an* $i \in I$ *such that* $\{u, v\} \subseteq X_i$; *(3) For all* $i, j, k \in I$, *if* j *is on the unique path from* i *to* k *in* T, *then we have:* $X_i \cap X_k \subseteq X_j$. *The* width *of* $\langle \{X_i \mid i \in I\}, T\rangle$ *is the* $max_{i \in I}|X_i| - 1$. *The* tree-width *of* G *is defined as the minimum width over all tree decompositions. The nodes of the tree* T *are called* T-nodes *and each* X_i *is called a* bag.

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Definition 2. *Given a graph $G = (V, E)$ and a set $\Pi \subseteq V$, the subset $R \subseteq V$ is called Π-*strong region *if $R \not\subseteq \mathcal{P}(\Pi \cup nbr(R))$, otherwise the set R is called Π-*weak region. *The region R is called* minimal Π-strong *if it is a Π-strong region and $\forall r \in R$, $R - r$ is a Π-weak region.*

Approximation Algorithm

Algorithm 1. $(k+1)$-approximation Algorithm

Given a tree decomposition $\langle \{X_i | i \in I\}, T \rangle$, take an arbitrary node, r, as a root of T.
Let I_ℓ be the set of T-nodes of distance ℓ from the root, and let d be the maximum distance from r.
$\Pi \leftarrow \emptyset, \quad a \leftarrow 0$
for $i = d$ to 0 **do**
    Let $I_i = \{r_1, \ldots, r_k\}$ and let T_j be the subtree in T rooted at r_j.
    Let Y_j be the set of nodes in G corresponding to the T-nodes in T_j.
    for all induced subgraph $G_j = G[Y_j]$ **do**
        if G_j is Π-strong **then**
            $\Pi \leftarrow \Pi \cup X_{r_j}$, $a \leftarrow a+1$, $ST_a \leftarrow Y_j \setminus \bigcup_{s=1}^{a-1} ST_s$; where ST_a is the a-th strong region found.
        end if
    end for
end for
Output $\Pi_O = \Pi$

PTAS for ℓ-round PDS on Planar Graphs

The idea is to use dynamic programming based on the following proposition.

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Proposition 1. *Given a pair* $\langle G, V' \rangle$ *where* $G = (V, E)$ *is a planar graph with tree-width* k *and* $V' \subseteq V$*, a minimum size set* $S \subseteq V$ *such that* $V' \subseteq \mathcal{P}^{\ell}(S)$ *can be obtained in time* $O(c^{k \log \ell} \cdot |V|)$*, for a constant* c*.*

PTAS Algorithm

Algorithm 2. PTAS for ℓ-round PDS

Given a planar embedding of G, and the parameter $0 < \epsilon \leq 1$.
Let $k = 4 \cdot \lceil \frac{\ell}{\epsilon} \rceil$.
for $i = 1$ to k **do**
  for all $j \geq 0$ **do**
    Solve "generalized" ℓ-round PDS on $\langle G[B_{i,j}], C_{i,j} \rangle$
    Let $\mathcal{O}_{i,j}$ be an optimal solution for $\langle G[B_{i,j}], C_{i,j} \rangle$
  end for
  $\mathit{\Pi}_i = \cup_{j \geq 0} \mathcal{O}_{i,j}$
end for
$r \leftarrow argmin\,\{|\mathit{\Pi}_i| : i = 1, \cdots, k\}$
Output $\mathit{\Pi}_O = \mathit{\Pi}_r$.

PDS in Directed Graphs

Theorem

The directed PDS problem even when restricted to directed acyclic graph cannot be approximated within ratio $2^{\log^{1-\epsilon} n}$, for any $\epsilon > 0$, unless $NP \subseteq DTIME\left(n^{poly \log n}\right)$.