1. Let $\mathcal{X}$ be an infinite set. For $p, q \in \mathcal{X}$, define

$$d(p, q) = \begin{cases} 1, & \text{if } p \neq q, \\ 0, & \text{if } p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

2. Show that the function $f(x) = x^n$ is uniformly continuous on $[-1, 1]$ for all $n \in \mathbb{Z}_+$. You can use the fact that,

$$x^n - y^n = (x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}, \quad \forall n \in \mathbb{Z}_+. \quad (1)$$

3. Let $\mathcal{U}$ and $\mathcal{W}$ represent subspaces of a linear space $\mathcal{V}$. Show that if every matrix in $\mathcal{V}$ belongs to $\mathcal{U}$ or $\mathcal{W}$, then $\mathcal{U} = \mathcal{V}$ or $\mathcal{W} = \mathcal{V}$.

4. Let $V$ be a Vector space, $S$ a Set, and $s \in S$. Let $U = \{f|f:S \to V\}$ and $W = \{f \in U|f(s) = 0\}$. Is $W$ a subspace of $U$?

5. Let $V$ be a vector space and let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of $V$. Suppose that $S$ is linearly independent. For an element $v$ of $V$, show that $v$ is not in the span of $S$ if and only if $\{v_1, v_2, \ldots, v_n, v\}$ is linearly independent.

6. Consider $X = [0, 2] \setminus \{1\}$ as a subspace of the real line $\mathbb{R}$. Show that the subset $[0, 1) \subset X$ is both open and closed in $X$.

7. (a) Is the set of rational numbers countable? Prove your answer statement.
(b) Repeat the above for irrational numbers and prove your answer statement.

8. Let $I = [0, 1]$ be the closed unit interval. Suppose $f$ is a continuous mapping of $I$ into $I$. Prove that $f(x) = x$ for at least one $x \in I$. 

9.

If \( Z \) is the field of complex numbers, which vectors in \( Z^3 \) are linear combinations of \((1, 0, -1)\), \((0, 1, 1)\) and \((1, 1, 1)\)?