COT 5615/ CIS 4930

Homework 1

1. Let $N$ be the set of natural numbers. Using Peano axioms,
   (a) Prove the cancellation law of addition, i.e., if $m, n, p \in N$ and $m + p = n + p$, then $m = n$.
   (b) Prove that multiplication is commutative, i.e., $n \cdot m = m \cdot n$, $\forall m, n \in N$.

2. Let $\mathcal{F}$ be an ordered field. Using only “field axioms”, show that the identity element of $\mathcal{F}$ is positive, i.e., $\mathcal{F} \ni 1 > 0$.

3. Let $\mathcal{S}$ be an ordered set, of which $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets, each bounded above and below.
   (a) If possible, relate $\sup (\mathcal{A} \cup \mathcal{B})$ to $\sup \mathcal{A}$ and $\sup \mathcal{B}$.
   (c) In the special case that $\mathcal{A} \subseteq \mathcal{B}$, relate $\inf \mathcal{A}$ and $\inf \mathcal{B}$.

4. Let $\mathcal{A}$ be a nonempty set of real numbers which is bounded below. Let $-\mathcal{A}$ be the set of all numbers $-x$, where $x \in \mathcal{A}$. Prove that $\inf \mathcal{A} = - (\sup (-\mathcal{A}))$.

5. Prove the theorem ‘Every non-zero element of $N$ has a predecessor in $N$.

6. Prove the following statements:
   (a) If $x \neq 0$ and $xy = xz$, then $y = z$.
   (b) If $x \neq 0$ and $xy = x$, then $y = 1$.
   (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.
   (d) If $x \neq 0$, then $1/(1/x) = x$.

7. If $z_1, \cdots, z_n$ are complex, prove that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.