Computational Geometry

Lecture 11: Arrangements and Duality
Question: In a set of $n$ points, are there 3 points on a line?
Three Points on a Line

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Faster algorithm: uses duality and arrangements

Note: other motivation in chapter 8 of the book
Duality

\[ \ell : y = mx + b \]

\[ p = (px, py) \]

Note:
Duality

\begin{align*}
\text{primal plane} & \quad \ell : y = mx + b \\
\text{dual plane} & \quad p^* : y = p_x x - p_y
\end{align*}

- \( p = (p_x, p_y) \)
- \( \ell^* = (m, -b) \)

point \( p = (p_x, p_y) \mapsto \) line \( p^* : y = p_x x - p_y \)
line \( \ell : y = mx + b \mapsto \) point \( \ell^* = (mx, -b) \)

Note:
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primal plane

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\[ p = (p_x, p_y) \]

\[ \text{point } p = (p_x, p_y) \mapsto \text{line } p^* : y = p_x x - p_y \]

line \[ \ell : y = mx + b \mapsto \text{point } \ell^* = (mx, -b) \]

\[ \text{Note: self inverse } (p^*)^* = p, \quad (\ell^*)^* = \ell \]
Duality

**Primal Plane**
\[ \ell : y = mx + b \]

**Dual Plane**
\[ p^* : y = p_x x - p_y \]

- **Point** \( p = (p_x, p_y) \) maps to **Line** \( \ell^* = (m, -b) \)

**Line** \( \ell : y = mx + b \) maps to **Point** \( p^* = (mx, -b) \)

*Note:* does not handle vertical lines
Duality preserves vertical distances
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⇒ incidence preserving: \( p \in \ell \) if and only if \( \ell^* \in p^* \)
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⇒ incidence preserving: \( p \in \ell \) if and only if \( \ell^* \in p^* \)
⇒ order preserving: \( p \) lies above \( \ell \) if and only if \( \ell^* \) lies above \( p^* \)
It can be applied to other objects, like segments.

primal plane

The dual of a segment is a double wedge.

Question: What line would dualize to a point in the right part of the double wedge?
It can be applied to other objects, like segments

The dual of a segment is a double wedge

**Question:** What line would dualize to a point in the right part of the double wedge?
A geometric interpretation:

- parabola $\mathcal{U} : y = x^2 / 2$
- point $p = (p_x, p_y)$ on $\mathcal{U}$
- derivative of $\mathcal{U}$ at $p$ is $p_x$, i.e., $p^*$ has same slope as the tangent line
- the tangent line intersects $y$-axis at $(0, -p_x^2/2)$
- $\Rightarrow p^*$ is the tangent line at $p$
A geometric interpretation:

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- \( \Rightarrow \) \( p^* \) is the tangent line at \( p \)
Usefulness of Duality

Why use duality? It gives a new perspective!

Detecting three points on a line dualizes to detecting three lines intersecting in a point
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Next we use arrangements
Arrangement $\mathcal{A}(L)$: subdivision induced by a set of lines $L$

- consists of *faces*, *edges* and *vertices* (some unbounded)
- arrangements exist of other geometric objects too, like line segments, circles, higher-dimensional objects
Arrangements of Lines

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Arrangements of Lines

Combinatorial Complexity:

- $\leq n(n - 1)/2$ vertices
- $\leq n^2$ edges
- $\leq n^2/2 + n/2 + 1$ faces:
  add lines incrementally
  \[ 1 + \sum_{i=1}^{n} i = n(n + 1)/2 + 1 \]

- equality holds in *simple* arrangements
Arrangements of Lines

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Overall $O(n^2)$ complexity
**Goal:** Compute $\mathcal{A}(L)$ in bounding box in DCEL representation

- plane sweep for line segment intersection:
  $O((n + k) \log n) = O(n^2 \log n)$
- faster: incremental construction
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Algorithm \textsc{ConstructArrangement}(L)

\textit{Input.} Set \( L \) of \( n \) lines
\textit{Output.} DCEL for \( \mathcal{A}(L) \) in \( \mathcal{B}(L) \)
1. Compute bounding box \( \mathcal{B}(L) \)
2. Construct DCEL for subdivision induced by \( \mathcal{B}(L) \)
3. \textbf{for} \( i \leftarrow 1 \) \textbf{to} \( n \)
4. \textbf{do} insert \( \ell_i \)
Incremental Construction

**Algorithm** `CONSTRUCTARRANGEMENT(L)`

*Input.* A set $L$ of $n$ lines in the plane

*Output.* DCEL for subdivision induced by $B(L)$ and the part of $A(L)$ inside $B(L)$, where $B(L)$ is a suitable bounding box

1. Compute a bounding box $B(L)$ that contains all vertices of $A(L)$ in its interior
2. Construct DCEL for the subdivision induced by $B(L)$
3. for $i \leftarrow 1$ to $n$
   4. do Find the edge $e$ on $B(L)$ that contains the leftmost intersection point of $\ell_i$ and $A_i$
   5. $f \leftarrow$ the bounded face incident to $e$
   6. while $f$ is not the unbounded face, that is, the face outside $B(L)$
   7. do Split $f$, and set $f$ to be the next face intersected by $\ell_i$
Face split:

\[ \ell_i \Rightarrow f \]

\[ f \]

\[ \ell_i \]

\[ \Rightarrow \]
Runtime analysis:

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1. $O(n^2)$

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2. constant

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3. & 4. ?

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The zone of a line $\ell$ in an arrangement $\mathcal{A}(L)$ is the set of faces of $\mathcal{A}(L)$ whose closure intersects $\ell$. 

**Zone Theorem:**

The complexity of the zone of a line in an arrangement of $m$ lines is $O(m)$. 
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**Theorem:** The complexity of the zone of a line in an arrangement of $m$ lines is $O(m)$.
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**Proof:**
- We can assume \( \ell \) horizontal and no other line is horizontal.
- We count number of *left-bounding* edges.
- We show by induction on \( m \) that this at most \( 5m \):
  - \( m = 1 \): trivially true.
  - \( m > 1 \): only at most 3 new edges if \( \ell \) is unique.
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- We count number of *left-bounding* edges.
- We show by induction on \( m \) that this at most \( 5m \):
  - \( m = 1 \): trivially true
  - \( m > 1 \): only at most 3 new edges if \( \ell_1 \) is unique, at most 5 if \( \ell_1 \) is not unique
  
  \[ 5(m - 1) + 5 = 5m \]
Run time analysis:

**Algorithm** `CONSTRUCT_ARRANGEMENT(L)`

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*Output.* DCEL for $\mathcal{A}(L)$ in $\mathcal{B}(L)$.

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Computational Geometry

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Incremental Construction

Run time analysis:

1. $O(n^2)$

Algorithm $\text{CONSTRUCT.ARRANGEMENT}(L)$

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Output. DCEL for $A(L)$ in $B(L)$.

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Run time analysis:

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   \[ \sum_{i=1}^{n} O(i) = O(n^2) \]

Algorithm $\text{constructArrangement}(L)$

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Run time analysis:

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$\sum_{i=1}^{n} O(i) = O(n^2)$

In total $O(n^2)$

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\textit{Output.} DCEL for \textit{A}(L) in \textit{B}(L).

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3 Points on a Line

Algorithm:
run incremental construction algorithm for dual problem
stop when 3 lines pass through a point

Run time: $O(n^2)$
3 Points on a Line

Algorithm:
- run incremental construction algorithm for dual problem
- stop when 3 lines pass through a point

Run time: \(O(n^2)\)
Example: Motion Planning

Where can the rod move by translation (no rotations) while avoiding obstacles?

- pick a reference point: lower end-point of rod
- shrink rod to a point, expand obstacles accordingly: locus of semi-free placements
- reachable configurations: cell of initial configuration in arrangement of line segments
Example: Motion Planning

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k-levels in Arrangements

The level of a point in an arrangement of lines is the number of lines strictly above it.
k-levels in Arrangements

The **level** of a point in an arrangement of lines is the number of lines strictly above it.

**Open problem:** What is the complexity of k-levels?
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**Dual problem:** What is the number of k-sets in a point set?
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**Known bounds:**
- Erdös et al. '73: $\Omega(n \log k)$ and $O(n k^{1/2})$
- Dey '97: $O(n k^{1/3})$
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In 3D, we have point-plane duality; lines dualize to other lines.

An arrangement induced by $n$ planes in 3D has complexity $O(n^3)$.

Deciding whether a set of points in 3D has four or more co-planar points can be done in $O(n^3)$ time (dualize and construct the arrangement).
Duality is a useful tool to reformulate certain problems on points in the plane to lines in the plane, and vice versa.

Dualization of line segments is especially useful.

Arrangements, zones of lines in arrangements, and levels in arrangements are useful concepts in computational geometry.

All of this exists in three and higher dimensional spaces too.