1. (25 pts) Solve the following recurrence relation without using the master theorem. Your final solution should be in $O()$ notation.

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}.$$ Assume $T(0) = 0$.

2. (15+10 pts) Consider a binary Min-Heap as discussed in class, represented in an array $A[1...n]$.

   a. Suppose you need to change the value of its $i$th element. Give an efficient algorithm that sets $A[i] = x$ and restores the heap property.

   b. Analyze and give an asymptotic upper bound on the time complexity of your algorithm.

3. (25 pts) If a graph has a $k$-clique, it is clear that any coloring of the vertices (such that no two adjacent vertices are colored the same) must use at least $k$ colors. However, $k$ colors may not be sufficient. Give an example of a graph in which the largest clique size is $3$, but $4$ colors are needed to color the graph. You have to prove that the graph is not $3$-colorable.

4. (10+10+5 pts) Consider as input an $n \times n$ array of positive integers wrapped around a cylinder (so that the $n$th row is followed by the 1st row). A path across the cylinder is a succession of $n$ array elements, such that the first element in the path is an element of the first column, the last element in the path is an element of the last column, and an element can be followed only by one of the three adjacent elements in the next column, namely, $(i, j+1)$, $(i+1, j+1)$, or $(i+1, j+1)$ (where all arithmetic is done modulo $n$). The cost of a path is simply the sum of the values of the elements in the squares crossed on the path.

   a. Give a recurrence relation for Min-Cost$(i,j)$, the minimum cost of a path starting at the first column and reaching element $(i, j)$.

   b. Use the recurrence relation to design an efficient algorithm which finds a minimum-cost path across the cylinder.

   c. Analyze and give an asymptotic upper bound on the time complexity of your algorithm.