**Review Questions II**

**Problem 8 – Quadratures**

We have done this problem (and similar ones) in class many times. You should not have any difficulty working out the details. If not, try reading the supplementary notes first.

**Problem 9 – Quadratures**

Yes, it is true. The proof is simple. Since $Q$ is an interpolatory quadrature rule, $Q(1)$ should give the integral of the constant function $f = 1$ over the interval $[0, 1]$, which is 1. This implies that $\sum_{i=1}^{n} w_i = 1$.

**Problem 10 – Quadratures**

Given the interval $[a, b]$. Let $h$ denote the length of this interval. The two nodes in the open two-point Newton-Cotes quadrature rule are $a + \frac{h}{3}, a + \frac{2h}{3}$. Let $w_1, w_2$ denote the weights associated to $a, b$, respectively. The problem can be solved easily by noting that $w_1 + w_2 = 1$ and $w_1 = w_2$. The former is clear while the latter follows from the usual symmetry argument (i.e., $a$ and $b$ in a sense are indistinguishable because they have the same distance to the boundary). From this, it is obvious that $w_1 = w_2 = 0.5$.

The remaining problem of determining the order of this quadrature rule is for you to complete. The most straightforward way is to compute $Q(x^k)$ and compare the result with $\int_a^b x^k dx$.

**Problem 11 – Linear Least Squares**

Consider the vector $a$ as an $n \times 1$ matrix.

1. The $Q$ matrix is $\frac{a}{|a|}$ and the $R$ matrix is simply $|a|$.

2. We have $QRx \approx b$, where $Q, R$ are as above. This implies that

   $$x = R^{-1}Q^t b = \frac{1}{|a|^2} a^t b.$$

**Problem 12 – Linear Least Squares**

1. Note that since $B$ is orthogonal, any two columns of $B$, considered as two vectors, are orthogonal. Now because $B$ is also triangular, this means that $B$ must be diagonal!

2. The diagonal entries of $B$ are either 1 or $-1$ because $B$ is orthogonal.

3. (sketch) Suppose $A$ has two QR factorizations, $A = Q_1R_1 = Q_2R_2$. This implies that

   $$Q_2^{-1}Q_1 = R_2R_1^{-1}.$$  

   The left hand side is an orthogonal matrix (because $Q_2^{-1}, Q_2, Q_1$ are orthogonal) and the right hand side is an upper-triangular matrix. From the above, the right hand side must be diagonal with entries 1 and $-1$. 

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Problem 13 – Linear Least Squares
The trick is that the residual vector $r$ has to be orthogonal to the columns of $A$ (why?). From this, it is obvious that $c$ is the only choice.

Problem 14 – Linear Least Squares

1. The Euclidean norm of the minimum residual vector is 1. Notice that you can put in any values for $x_1, x_2$ and you can never get the third component of $b = [2, 1, 1]^t$ correctly. The best you can do is to let $x_1 = x_2 = 1$ and this will give you $[2, 1]^t$ for the first two components. The residual vector is $[0, 0, 1]^t$ and its norm of course is 1.

2. $x_1 = x_2 = 1$ as said above.

Problem 15 – Linear Least Squares

1. To show that $H$ is orthogonal, you need to show that $H^t H = HH^t = I$. This is straightforward.

2. Use the formula $v = a - \alpha e_1$, where $a$ is the vector $[1, 1, 1, 1]^t$ and $\alpha$ is the magnitude of $a$, which is 2. It turns out that $v = [-1, 1, 1, 1]^t$ and $\alpha = 2$.

Problem 16 – ODEs
Consider the initial value problem (IVP)

$$y'' = y$$

for $t \geq 0$, with initial values $y(0) = 1$ and $y'(0) = 2$.

1. Let $Y = \begin{bmatrix} y \\ y' \end{bmatrix}$ and the ODE $y'' = y$ got transformed into the equivalent linear system

$$\frac{dY}{dt} = AY,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2. The initial condition for Part a) is $Y(0) = [1, 2]^t$.

3. The solutions are not stable because $A$ has 1 (the other is $-1$) as an eigenvalue, which is $> 0$.

4. The formula is $Y(k + 1) = Y(k) + hf(x, y)$. With this, we have $Y(1) = Y(0) + 0.5AY(0) = [2, 2.5]^t$.

5. No.

6. No.
Problem 17 – ODEs

As before, we will use the linear ODE \( y' = \lambda y \) to analyze the accuracy and stability of each method. The general solution to this ODE is \( y = Ce^{\lambda t} \). Given a step size \( h \), the real solution behaves according to \( y_{k+1} = e^{h\lambda}y_k \). The growth factor is \( e^{h\lambda} = 1 + h\lambda + \frac{h^2\lambda^2}{2} + \cdots \), using Taylor expansion.

Part a): We have \( y_{k+1} = y_k + h\lambda \frac{y_k + y_{k-1}}{2} \).

\[
(1 - \frac{h\lambda}{2})y_{k+1} = (1 + \frac{h\lambda}{2})y_k.
\]

This gives

\[
y_{k+1} = \frac{(1 + \frac{h\lambda}{2})}{1 - \frac{h\lambda}{2}} y_k.
\]

Recall that

\[
\frac{1}{1 - \frac{h\lambda}{2}} = 1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{4} + \frac{h^3\lambda^3}{8} + \cdots.
\]

This implies that

\[
y_{k+1} = (1 + h\lambda + \frac{h^2\lambda^2}{2} + \cdots) y_k.
\]

Comparing with above, we see that the accuracy is of first-order. Furthermore, the growth factor must have absolute value less than 1:

\[
|\frac{(1 + \frac{h\lambda}{2})}{1 - \frac{h\lambda}{2}}| < 1,
\]

which holds for any \( h > 0 \) when \( \lambda < 0 \).

Part b): For the two-step leapfrog method, we have \( y_{k+1} = y_{k-1} + 2h\lambda y_k \). OK, this problem is little trickier than the others. What we need to figure out is the multiplicative factor \( \mu \)

\[
y_{k+1} = \mu y_k
\]

for \( k \) sufficiently large. Starting with \( y_{k+1} = y_{k-1} + 2h\lambda y_k \), we have

\[
\frac{y_{k+1}}{y_k} = \frac{y_{k-1}}{y_k} + 2h\lambda.
\]

For \( k \) sufficiently large, the above equation got translated into

\[
\mu = \frac{1}{\mu} + 2h\lambda,
\]

or \( \mu^2 - 2h\lambda\mu - 1 = 0 \). This implies that

\[
\mu = h\lambda + \sqrt{1 + h^2\lambda^2}.
\]

The other solution \( \mu = h\lambda - \sqrt{1 + h^2\lambda^2} \) is clearly inappropriate since it is negative. With \( \mu \), we can figure out the accuracy and the stable range for \( \lambda \). First, we note that using Taylor expansion

\[
\sqrt{1 + h^2\lambda^2} = 1 + \frac{1}{2}h^2\lambda^2 - \frac{1}{4}h^4\lambda^4 + \cdots.
\]
Therefore, we have \( \mu = h\lambda + \sqrt{1 + h^2\lambda^2} = 1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \cdots \). This shows that the accuracy is second-order. Finally, because \( |h\lambda| < \sqrt{1 + h^2\lambda^2} \) (why?), \( |\mu| < 1 \) whenever \( \lambda < 0 \). Therefore, as long as \( \lambda < 0 \), there is no restriction on the step size \( h > 0 \).

**Problem 18 – Quadratures**

1. Let’s show that \( p(x) \) has \( n \) real roots. The same idea (Yes! Exactly the same idea) can be applied to show that all roots of \( p(x) \) are simple and lie in the interval \((a, b)\). Here is the proof. Assume that \( p(x) \) has only \( m < n \) real roots, \( x_1, \ldots, x_m \). This means that \( p(x) = q(x)r(x) \) where \( q(x) \) is the product of \((x - x_1), \ldots, (x - x_m)\) and \( r(x) \) is some real polynomial, which is the product of \((x - y_1), \ldots, (x - y_l)\), where \( y_i \) are the complex roots of \( p(x) \). \( r(x) \) is a polynomial without real roots; therefore, we can assume that \( r(x) > 0 \) for all \( x \) (why?). Consider the integral

\[
\int_a^b p(x)q(x)dx.
\]

Notice that \( q(x) \) has degree \( < n \) and therefore, the integral above is zero according to the problem. However, the integral can also be expressed as

\[
\int_a^b p(x)q(x)dx = \int_a^b q^2(x)r(x)dx,
\]

where the integrand is non-negative. It follows that the integral has to be positive, which is a contradiction.

2. Let \( P(x) \) be any polynomial of degree \( < 2n \). \( P(x) = p(x)q(x) + r(x) \) for some polynomials \( q(x), r(x) \), where \( q(x), r(x) \) have degrees less than \( n \). The important point to observe is that

\[
\int_a^b P(x)dx = \int_a^b p(x)q(x) + r(x)dx = \int_a^b r(x)dx.
\]

However, since \( x_i \) are the roots of \( p(x) \), \( Q(P(x)) = \sum_{i=1}^n w_iP(x_i) = \sum_{i=1}^n w_ir(x_i) = Q(r(x)) \). Comparing with the above result, we have

\[
Q(P(x)) = Q(r(x)) = \int_a^b r(x)dx = \int_a^b P(x)dx.
\]

You have to figure out why \( Q(r(x)) = \int_a^b r(x)dx \). It follows that for every polynomial \( P(x) \) with degree \( < 2n \), \( Q(P(x)) = \int_a^b P(x)dx \). That is, \( Q \) is of order \( 2n \).

3. One such polynomial is \( 6x^2 - 6x + 1 \).

4. The two roots of the polynomial above are \( \frac{6 \pm \sqrt{12}}{12} \). You can easily verify that the two roots indeed lie in the interval \((0, 1)\). Let \( w_1, w_2 \) denote the weights associated to the two nodes. Clearly, \( w_1 + w_2 = 1 \) (Why?) and because of the symmetry, \( w_1 = w_2 \) (Why?). This tells you immediately that \( w_1 = w_2 = 0.5 \).