Section 3.2

19. (a) The significant terms here are the $n^2$ being multiplied by the $n$; thus this function is $O(n^3)$.
(b) since $\log n$ is smaller than $n$, the significant term in the first factor is $n^2$. Therefore the entire function is $O(n^5)$.
(c) For the first factor we note that $2^n < n!$ for $n \geq 4$, so the significant term is $n!$. For the second factor, the significant term is $n^3$. Therefore this function is $O(n^3 n!)$.

21. (a) First we note that $\log(n^2 + 1)$ and $\log n$ are in the same big-O class, since $\log n^2 = 2 \log n$. Therefore the second term here dominates the first, and the simplest good answer would be just $O(n^2 \log n)$.
(b) The first term is the same big-O class as $O(n^2(\log n)^2)$, while the second is in a slightly smaller class, $O(n^2 \log n)$. Therefore the answer is $O(n^2(\log n)^3)$.
(c) The only issue here is whether $2^n$ or $n^2$ is the faster-growing, and clearly it is the former. Therefore the best big-O estimate we can give is $O(n^{2^n})$.

26.

We just need to look at the definitions. To say that $f(x)$ is $O(g(x))$ means that there are constants $C$ and $k$ such that $|f(x)| \leq C|g(x)|$ for all $x > k$. Note that without loss of generality we may take $C$ and $k$ to be positive. To say that $g(x)$ is $\Omega(f(x))$ is to say that there are positive constants $C'$ and $k'$ such that $|g(x)| \geq C'|f(x)|$ for all $x > k$. These are saying exactly the same thing if we set $C' = 1/C$ and $k' = k$.

Section 4.1
4. 
(a) Plugging in \( n = 1 \) we have that \( P(1) \) is the statement \( 1^3 = \left[ 1 \cdot (1+1)/2 \right]^2 \).
(b) Both sides of \( P(1) \) shown in part (a) equal 1.
(c) The inductive hypothesis is the statement that
\[
1^3 + 2^3 + \cdots + k^3 = \left( \frac{k(k+1)}{2} \right)^2.
\]
(d) For the inductive step, we want to show for each \( k \geq 1 \) that \( P(k) \) implies \( P(k+1) \). In other words, we want to show that assuming the inductive hypothesis we can prove
\[
[1^3 + 2^3 + \cdots + k^3] + (k+1)^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2.
\]
(e) Replacing the quantity in brackets on the left-hand side of part (d) by what it equals by virtue of the inductive hypothesis, we have
\[
\left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3 = (k+1)^2 \left( \frac{k^2}{4} + k + 1 \right) = \left( \frac{(k+1)(k+2)}{2} \right)^2
\]
as desired.
(f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer \( n \).

10. 
(a) By computing the first few sums and getting the answers \( 1/2, 2/3, \) and \( 3/4 \), we guess that the sum is \( n/(n+1) \).
(b) We prove this by induction. It is clear for \( n = 1 \), since there is just one term, \( 1/2 \). Suppose that
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.
\]
We want to show that
\[
\left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} \right] + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.
\]
Starting from the left, we replace the quantity in brackets by \( k/(k + 1) \), and then do the algebra

\[
\frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} = \frac{k^2 + 2k + 1}{(k + 1)(k + 2)} = \frac{k + 1}{k + 2},
\]

yielding the desired expression.

16. The basis step reduces to \( 6 = 6 \). Assuming the inductive hypothesis we have

\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)
= \frac{k(k + 1)(k + 2)(k + 3)}{4} + (k + 1)(k + 2)(k + 3)
= (k + 1)(k + 2)(k + 3) \left( \frac{k}{4} + 1 \right)
= \frac{(k + 1)(k + 2)(k + 3)(k + 4)}{4}.
\]

20. The basis step is \( n = 7 \), and indeed \( 3^7 < 7! \), since \( 2187 < 5040 \). Assume the statement for \( k \). Then \( 3^{k+1} = 3 \cdot 3^k < (k + 1) \cdot 3^k < (k + 1) \cdot k! = (k + 1)! \), the statement for \( k + 1 \).

33. To prove that \( P(n) : 5\mid(n^5 - n) \) holds for all nonnegative integers \( n \), we first check that \( P(0) \) is true; indeed \( 5\mid0 \). Next assume that \( 5\mid(n^5 - n) \), so that we can write \( n^5 - n = 5t \) for some integer \( t \). Then we want to prove \( P(n+1) \), namely that \( 5\mid((n + 1)^5 - (n + 1)) \). We expand and then factor the right-hand side to obtain

\[
(n + 1)^5 - (n + 1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1
= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n)
= 5t + 5(n^4 + 2n^3 + 2n^2 + n)
= 5(t + n^4 + 2n^3 + 2n^2 + n).
\]

Thus we have shown that \((n + 1)^5 - (n + 1)\) is also a multiple of 5, and our proof by induction is complete.
The trick is to induct not on \( n \) itself, but rather on \( \log_2 n \). In other words, we write \( n = 2^k \) and prove the statement by induction on \( k \). This will prove the statement for every \( n \) that is a power of 2; a separate argument is needed to extend to the general case.

We take the basis step to be \( k = 1 \), so that \( n = 2^1 = 2 \). In this case the trick is to start with the true inequality \( (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0 \). Expanding, we have

\[
 a_1 - 2\sqrt{a_1a_2} + a_2 \geq 0,
\]

whence \( (a_1 + a_2)/2 \geq (a_1a_2)^{1/2} \), as desired. For the inductive step, we assume that the inequality holds for \( n = 2^k \) and prove that it also holds for \( 2n = 2^{k+1} \). What we need to show is that

\[
 \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} \geq (a_1a_2\cdots a_{2n})^{1/(2n)}.
\]

First we observe that

\[
 \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = \left( \frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n} \right) / 2
\]

and

\[
 (a_1a_2\cdots a_{2n})^{1/(2n)} = \left( (a_1a_2\cdots a_n)^{1/n}(a_{n+1}a_{n+2}\cdots a_{2n})^{1/n} \right)^{1/2}.
\]

Now to simplify notation, let \( A(x, y, \ldots) \) denote the arithmetic mean and \( G(x, y, \ldots) \) denote the geometric mean of the numbers \( x, y, \ldots \). It is clear that if \( x \leq x', y \leq y' \), and so on, then \( A(x, y, \ldots) \leq A(x', y', \ldots) \) and \( G(x, y, \ldots) \leq G(x', y', \ldots) \). Now we have

\[
 A(a_1, \ldots, a_{2n}) = A(A(a_1, \ldots, a_n), A(a_{n+1}, \ldots, a_{2n})) \\
 \geq A(G(a_1, \ldots, a_n), G(a_{n+1}, \ldots, a_{2n})) \\
 \geq G(G(a_1, \ldots, a_n), G(a_{n+1}, \ldots, a_{2n})) \\
 = G(a_1, \ldots, a_{2n})
\]

Having proved the inequality in the case in which \( n \) is a power of 2, we now turn to the case of \( n \) that is not a power of 2. Let \( m \) be the smallest power of 2 bigger than \( n \). Denote the arithmetic mean \( A(a_1, \ldots, a_n) \) by \( a \), and set \( a_{n+1} = a_{n+2} = \cdots = a_m \) all equal to \( a \). One effect of this is that then \( A(a_1, \ldots, a_m) = a \) Now we have

\[
 \left( \prod_{i=1}^{n} a_i \right)^{m-n} \leq A(a_1, \ldots, a_m)
\]
by the case we have already proved, since \( m \) is a power of \( 2 \). Using algebra on the left-hand side and the observation that \( A(a_1, \ldots, a_m) = a \) on the right, we obtain

\[
\left( \prod_{i=1}^{n} a_i \right)^{1/m} a^{1-n/m} \leq a
\]

or

\[
\left( \prod_{i=1}^{n} a_i \right)^{1/m} \leq a^{n/m}.
\]

Finally we raise both sides to the power \( m/n \) to give

\[
\left( \prod_{i=1}^{n} a_i \right)^{1/n} \leq a,
\]

as desired.

Section 4.2

5.

(a) We can form the following amounts of postage as indicated: 4 = 4,
8 = 4 + 4, 11 = 11, 12 = 4 + 4 + 4, 15 = 11 + 4, 16 = 4 + 4 + 4 + 4, 19 = 11 + 4 + 4,
20 = 4 + 4 + 4 + 4 + 4, 22 = 11 + 11, 23 = 11 + 4 + 4 + 4, 24 = 4 + 4 + 4 + 4 + 4 + 4,
26 = 11 + 11 + 4, 27 = 11 + 4 + 4 + 4 + 4, 28 = 4 + 4 + 4 + 4 + 4 + 4 + 4,
30 = 11 + 11 + 4 + 4, 31 = 11 + 4 + 4 + 4 + 4 + 4, 32 = 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4,
33 = 11 + 11 + 11. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form all amounts of postage greater than or equal to 30 cents using just 4-cent and 11-cent stamps.

(b) Let \( P(n) \) be the statement that we can form \( n \) cents of postage using just 4-cent and 11-cent stamps. We want to prove that \( P(n) \) is true for all \( n \geq 30 \). The basis step, \( n = 30 \), is handled above. Assume that we can form \( k \) cents of postage; we will show how to form \( k + 1 \) cents of postage. If the \( k \) cents included an 11-cent stamp, then replace it by three 4-cent stamps. Otherwise, \( k \) cents was formed from just 4-cent stamps. Because \( k \geq 30 \), there must be at least eight 4-cent stamps involved. Replace eight 4-cent stamps by three 11-cent stamps, and we have formed \( k + 1 \) cents in postage.

(c) \( P(n) \) is the same as in part (b). To prove that \( P(n) \) is true for all \( n \geq 30 \), we note for the basis step that from part (a), \( P(n) \) is true for \( n = 30, 31, 32, 33 \). Assume the inductive hypothesis, that \( P(j) \) is true for all \( j \) with \( 30 \leq j \leq k \), where \( k \) is a fixed integer greater than or equal to 33. we want to show that \( P(k+1) \) is
Because $k - 3 \geq 30$, we know that $P(k - 3)$ is true, that is, that we can form $k - 3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k + 1$ cents of postage, as desired. In this proof our inductive hypothesis included all values between 30 and $k$ inclusive, and that enabled us to jump back four steps to a value for which we knew how to form the desired postage.

7. We can form the following amounts of money as indicated: $2 = 2$, $4 = 2 + 2$, $5 = 5$, $6 = 2 + 2 + 2$. By having considered all the combinations, we know that the gaps in the list cannot be filled. We claim that we can form all amounts of money greater than or equal to 5 dollars. Let $P(n)$ be the statement that we can form $n$ dollars using just 2-dollar and 5-dollar bills. We want to prove that $P(n)$ is true for all $n \geq 5$. We already observed that the basis step is true for $n = 5$ and 6. Assume the inductive hypothesis, the $P(j)$ is true for all $j$ with $5 \leq j \leq k$, where $k$ is a fixed integer greater than or equal to 6. We want to show that $P(k + 1)$ is true. Because $k - 1 \geq 5$, we know that $P(k - 1)$ is true, that is we can form $k - 1$ dollars. Add another 2-dollar bill, and we have formed $k + 1$ dollars, as desired.

12. The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Indeed this is simply the representation of a number in binary form (base two). Assume the inductive hypothesis, that every positive integer up to $k$ can be written as a sum of distinct powers of 2. We must show that $k + 1$ can be written as a sum of distinct powers of 2. If $k + 1$ is odd, then $k$ is even, so $2^0$ was not part of the sum for $k$. Therefore the sum for $k + 1$ is the same as the sum for $k$ with the extra term $2^0$ added. If $k + 1$ is even, then $(k + 1)/2$ is a positive integer, so by the inductive hypothesis $(k + 1)/2$ can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for $k + 1$.

26. (a) Clearly, these conditions tell us that $P(n)$ is true for the even values of $n$, namely, 0, 2, 4, 6, . . . Also, it is clear that there is no way to be sure that $P(n)$ is true for other values of $n$.

(b) Clearly, these conditions tell us that $P(n)$ is true for the values of $n$ that are multiples of 3, namely, 0, 3, 6, 9, . . . Also it is clear that there is no way to be sure that $P(n)$ is true for other values of $n$.

(c) These conditions are sufficient to prove by induction that $P(n)$ is true for
all nonnegative integers \( n \).

(d) We immediately know that \( P(0) \), \( P(2) \), and \( P(3) \) are true, and clearly there is no way to be sure that \( P(1) \) is true. Once we have \( P(2) \) and \( P(3) \), the inductive step \( P(n) \to P(n + 2) \) gives us the truth of \( P(n) \) for all \( n \geq 2 \).

29. The error is in going from the basis step \( n = 0 \) to the next value \( n = 1 \). We cannot write 1 as the sum of two smaller natural numbers, so we cannot invoke the inductive hypothesis. In the notation of the proof, when \( k = 0 \), we cannot write \( 0 + 1 = i + j \) where \( 0 \leq i \leq 0 \) and \( 0 \leq j \leq 0 \).

30. (Bonus)

The flaw comes in the inductive step, where we are implicitly assuming that \( k \geq 1 \) in order to talk about \( a^{k-1} \) in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). Our basis step was \( n = 0 \), so we are not justified in assuming that \( k \geq 1 \) when we try to prove the statement for \( k + 1 \) in the inductive step. Indeed, it is precisely at \( n = 1 \) that the proposition breaks down.

32.

The proof is invalid for \( k = 4 \). We cannot increase the postage from 4 cents to 5 cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that \( j \geq 3 \) is not mentioned.