COT3100 Discrete Structures and Applications

Midterm Exam 1

Nam Nguyen

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"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

In order for your test to be valid, please sign below to honor the code.

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Oct 1st, 2012
The exam time is 60 minutes.

**Question 1  [20 points]**

Please answer BOTH the following questions.

1. [10 points] Show that \((p \rightarrow q) \lor (p \rightarrow r)\) and \(p \rightarrow (q \lor r)\) are logically equivalent by a series of logical equivalences.

   We determine exactly which rows of the truth table will have T as their entries. Now \((p \rightarrow q) \lor (p \rightarrow r)\) will be true when either of the conditional statements is true. The conditional statement will be true if \(p\) is false, or if \(q\) in one case or \(r\) in the other case is true, i.e., when \(q \lor r\) is true, which is precisely when \(p \rightarrow (q \lor r)\) is true. Since the two propositions are true in exactly the same situations, they are logically equivalent.

2. [10 points] Determine whether the following argument is valid or not.

\[
\begin{align*}
p & \rightarrow r \\
q & \rightarrow r \\
q \lor \neg r & \\
\therefore \neg p 
\end{align*}
\]

It is invalid because when \(p, q, r\) are all true, all premises are true but the conclusion is false.
Question 2 [20 points]

Show that these statements about the real number $x$ are equivalent:

(i) $x$ is rational;
(ii) $x/2$ is rational;
(iii) $3x - 1$ is rational.

We give direct proofs that (i) implies (ii), that (ii) implies (iii), and that (iii) implies (i). That will suffice.

(i) $\rightarrow$ (ii): suppose that $x = p/q$ where $p$ and $q$ are integers with $q \neq 0$. Then $x/2 = p/(2q)$, and this is rational, since $p$ and $2q$ are integers with $2q \neq 0$.

(ii) $\rightarrow$ (iii): suppose that $x/2 = p/q$ where $p$ and $q$ are integers with $q \neq 0$. Then $x = (2p)/q$, so $3x - 1 = (6p)/q - 1 = (6p - q)/q$ and this is rational, since $6p - q$ and $q$ are integers with $q \neq 0$.

(iii) $\rightarrow$ (i): suppose that $3x - 1 = p/q$ where $p$ and $q$ are integers with $q \neq 0$. Then $x = (p/q + 1)/3 = (p + q)/(3q)$, and this is rational, since $p + q$ and $3q$ are integers with $3q \neq 0$. 
Question 3 [20 points]

Please answer EITHER ONE of the following two questions. No extra point will be given for answering both of them, and only the best score will be counted to the total.

(1) [20 points] Prove that there are no solutions in integers \( x \) and \( y \) to the equation \( 2x^2 + 5y^2 = 14 \).

If \( |y| \geq 2 \), then \( 2x^2 + 5y^2 \geq 2x^2 + 20 \geq 20 \), so the only possible values of \( y \) to try are 0 and \( \pm 1 \). In the former case we would be looking for solutions to \( 2x^2 = 14 \) and in the latter case to \( 2x^2 = 9 \). Clearly there are no integer solutions to these equations, so there are no solutions to the original equation.
(2) [20 points] Prove that if $x^3$ is irrational, then $x$ is irrational.

A proof by contraposition: If $x$ is rational, then $x = p/q$ for some integers $p$ and $q$ with $q \neq 0$. Then $x^3 = p^3/q^3$, and we have expressed $x^3$ as the quotient of two integers, the second of which is not zero. This by definition means that $x^3$ is rational, and that completes the proof of the contrapositive of the original statement.
Question 4  [20 points]

Show that (NOT by using Venn diagram) if $A$ and $B$ are sets, then $A - B = A \cap \overline{B}$.

Both sides equal to $\{x | x \in A \land x \notin B\}$
Question 5  [20 points]

Please answer all three following questions.

Let $A = \{3, 5, 6\}$, $B = \{a, b\}$.


$A[2]$ is unknown since there is no order in a set.

(2) [8 points] Find $A \times B$.

$\{(3, a), (3, b), (5, a), (5, b), (6, a), (6, b)\}$

(3) [8 points] Find $P(A)$.

$P(A) = \{\emptyset, \{3\}, \{5\}, \{6\}, \{3, 5\}, \{3, 6\}, \{5, 6\}, \{3, 5, 6\}\}$
Bonus Question [3 extra points for course total]

Prove that all the solutions to the equation $x^2 = x + 1$ are irrational.

Suppose the equation has a rational solution $p/q$ where $p$ and $q$ are coprime, and $q \neq 0$. Then we have
\[
\frac{p^2}{q^2} - \frac{p}{q} = 1,
\]
so $p(p - q) = q^2$, which means $p$ divides $q^2$, which is contradiction to the assumption that $p$ and $q$ are coprime.