Equivalence of Propositions

1. Truth tables: two same columns

2. Sequence of logical equivalences: from one to the other using equivalence laws
Equivalence laws

Table 6 & 7 in §1.2, some often used:

**Associative:**

\[(p \lor q) \lor r \equiv p \lor (q \lor r),\ (p \land q) \land r \equiv p \land (q \land r)\]

**Distributive:**

\[p \lor (q \land r) \equiv (p \lor q) \land (p \lor r),\ p \land (q \lor r) \equiv (p \land q) \lor (p \land r)\]
Equivalence laws

Table 6 & 7 in §1.2, some often used:

**De Morgan’s:**

\[
\neg (p \land q) \equiv \neg p \lor \neg q, \quad \neg (p \lor q) \equiv \neg p \land \neg q
\]

**Conditionals:**

\[
p \rightarrow q \equiv \neg p \lor q, \quad p \rightarrow q \equiv \neg q \rightarrow \neg p
\]
Tautology

1. all T column in the truth table;
2. apply equivalences till T.
Predicate, Quantifiers

**Predicate**: proposition with variable(s), $P(x)$

**Quantifier**: $\forall x P(x)$, $\exists x P(x)$

**Negation**: 

$\neg \forall x P(x) \equiv \exists x \neg P(x)$, $\neg \exists x Q(x) \equiv \forall x \neg Q(x)$

$\neg \forall x \exists y \forall z P(x, y, z) \equiv \exists x \forall y \exists z \neg P(x, y, z)$
Rules of inference

All premises are True;
apply rules of inference to show the conclusion True;

Rules: understand Tables 1 & 2 of §1.6
Proofs

Terminology: proof, theorem, axiom, lemma, corollary, conjecture

Methods of proving $p \rightarrow q$:

- **Direct proof**: assume $p$ True, find out $q$ True;
- **Contraposition**: directly prove $\neg q \rightarrow \neg p$;
- **Contradiction**: assume $p$ True and $q$ False, prove $\neg q \rightarrow \neg p$, then $p$ and $\neg p$ contradict.
Strategies of proofs

Proof by **cases**: cases cover all possibilities

example: prove \( \max(x, y) + \min(x, y) = x + y \)

cases \( x \geq y; \ x < y \)

**exhaustive**: check each possible input

example: \( (n+1)^3 \geq 3^n \) if \( n \) is a positive integer with \( n \leq 4 \).
Strategies of proofs

Existence proof of $\exists x P(x)$:

**constructive**: find $a$ such that $P(a)$ True

**nonconstructive**: not finding $a$
Strategies of proofs

disproving $\forall x P(x)$:

**constructive**: find $a$ such that $P(a)$ False;
$\neg \forall x P(x) \equiv \exists x \neg P(x)$

**nonconstructive**: not finding $a$
Sets

Definitions: set (unique, unordered elements); subset $\subseteq$; proper subset $\subset$; size/cardinality $|A|$; power set; cartesian product;

What is the cardinality of power set of $A$?

What is the cardinality of $A \times B$?
Set Operations

Definitions:

union $A \cup B$; intersection $A \cap B$; disjointness of $A$ and $B$; difference $A-B$; universal set $U$; complement of $A$
Set Identities

Understand Table 1 of §2.2; note the Distributive laws and De Morgan’s laws because they connect $\cap$ and $\cup$.

$A - B = A \cap (\neg B)$ connects $\cap$ and $\neg$.
Proof of Set Identities

Membership table: similar to truth table;

Applying set identity laws.

One common approach of proving $A = B$: prove $A \subseteq B$ and $B \subseteq A$
Functions

Definition of function: domain, codomain, mapping rule

- Can the domain have elements not get mapped?
- Can the domain have elements that mapped to multiple elements in the codomain?
- What is the differences between codomain and range of a function?
Functions

How to check if a function is one-to-one?
How to check if a function is onto?
On what condition a function has an inverse?
If a function is not one-to-one, why doesn’t it have an inverse function?
If a function is not onto, why doesn’t it have an inverse function?
Functions and cardinalities

\( f: A \rightarrow B, \) \( A \) and \( B \) have finite elements

\( f \) one-to-one \quad |A| \leq |B|

\( f \) onto \quad |A| \geq |B|

\( f \) bijection \quad |A| = |B|

Sequences and Summations

- definitions of sequence \( \{a_n\} \): 
  by a formula; by recurrence

- summation of sequence \( \{a_n\} \): 
  be careful on the starting and ending subscripts of the summation; should be clear on thing like

\[
\sum_{j=1}^{n} \sum_{i=j}^{n} \]


Complexity

Definition of $f(x) = O(g(x))$:

$$f(x) \leq Cg(x) \; \forall x \geq k$$

- only need to find one pair of $(C, k)$
- can fix/pick one first, then find the other
Complexity

- Definition of $O(g(x))$, $\Omega(g(x))$, $\Theta(g(x))$

- How to estimate the complexity of the sum of two functions?

- How to estimate the complexity of the product of two functions?

- If $a > 1$, then the complexity: $a < \log \log n < \log n < n^{1/a} < n < n^a < a^n < n!$
Algorithm

Design your algorithm and estimate the complexity

• make sure it is correct;

• no matter if it is correct, the complexity can be estimated
Induction

Mathematical induction: $P(k) \rightarrow P(k+1)$

to prove $P(k+1)$, only need $P(k)$

Strong induction: $P(1) \land \ldots \land P(k) \rightarrow P(k+1)$

to prove $P(k+1)$, need all previous established cases
Counting

**Product rule**: the task in different steps; every step is necessary

**Sum rule**: the task in different categories; every category is mutually exclusive

- Often use sum rule to split the problem into subproblems, and use product rule in each of them
- Hard part is often the sum rule; keep them mutually exclusive to avoid recounting or missing
Pigeonhole Principle

Hard part is how to build/select pigeonholes/categories; make sure to do enough exercises to familiarize
Permutation and Combination

the order of the selected elements matters in permutation; which is not in combination their connection:

\[ P(n,r) = \frac{n!}{(n-r)!}, \quad C(n,r) = \frac{n!}{r!(n-r)!} \]

\[ P(n,r) = r! C(n,r) = P(r,r) C(n,r) \]

You can pick \( r \) elements from \( n \) (which is \( C(n,r) \)) first, and permute the \( r \) elements to get \( P(n,r) \).
Binomial Theorem

\[(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n\]