Linear Shift Invariant Systems (LSI)

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Linear Shift Invariant Systems

1. Image Processing (Chapter 6 from BKP Horn)
Key task in Image Processing: Transform an image to a form more amenable to further manipulation.

Imaging systems or image formation systems can be approximated by Linear Shift Invariant Systems (LSI), a powerful analytic tool.

Consider a 2D system with inputs $f_1(x, y)$ and $f_2(x, y)$ and outputs $g_1(x, y)$ and $g_2(x, y)$. 
System is linear if $Input = \alpha f_1 + \beta f_2$ and $Output = \alpha g_1 + \beta g_2$.

Which of the following systems are linear? (a) $g(x) = e^{\pi} f(x)$, (b) $g(x) = f(x) + 1$, (c) $g(x) = xf(x)$, (d) $g(x) = (f(x))^2$

**Shift Invariant**: $Input = f(x - a, y - b)$, $Output = g(x - a, y - b)$, for arbitrary $a, b$. In practice, images are limited in area and so, shift invariance holds only for limited areas.
Why are LSIs important?

- Can model image formation systems using LSIs
- System shortcomings can be discussed in terms of the LSI model.
- Study of LSI leads to useful algorithms for processing images digitally or even optically.
Consider a system with $f(x, y)$ as input that produces,

$$
g(x, y) = \int \int_{-\infty}^{\infty} f(x - \xi, y - \eta) h(\xi, \eta) d\xi d\eta \quad (1)
$$

i.e. $g_{\text{out}} = f_{\text{in}} \otimes h_{\text{in}}$ (convolution integral) \quad (2)

This system is an LSI system:

Input : $\alpha f_1 + \beta f_2$; Output : $\alpha g_1 + \beta g_2$

Input : $f(x - a, y - b)$; Output : $g(x - a, y - b)$

Thus, a system whose response is described by a convolution, $\otimes$ is an LSI and conversely, any LSI system performs a convolution.
Convolution Example

**Figure:** Convolution Example

Review of convolution

- Illustration of $h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$

original functions

$g(x-u)$, reversed and shifted to $x$

curve = product of $f(u)g(x-u)$

area = integral of $f(u)g(x-u)$

= value of $h()$ at $x$
**Impulse Functions**

**Question**: Given an arbitrary function $h(x, y)$, can we find a function $f(x, y)$ that causes the output to be $h(x, y)$ i.e.,

$$h(x, y) = \int\int_{-\infty}^{\infty} f(x - \xi, y - \eta) h(\xi, \eta) d\xi d\eta ? \quad (3)$$

**Ans**: Yes; called an impulse function.

$$\delta(x, y) = 0 \text{ if } (x, y) \neq 0$$

$$\int\int_{-\infty}^{\infty} \delta(x, y) = 1$$
Further, \( \delta(x, y) \otimes f(x, y) = f(x, y) \otimes \delta(x, y) = f(x, y) \).

\( \delta(ax, by) = (1/|ab|)\delta(x, y) \)
Sampling and Replicating Property

Consider an infinite sequence of impulses denoted by the “comb” or “shah” symbol,

$$\text{Ш}(x, y) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \delta(x - n, y - m)$$  \hspace{1cm} (4)

Properties:

$$\text{Ш}(ax, by) = \frac{1}{|ab|} \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \delta(x - n/a, y - m/b)$$  \hspace{1cm} (5)

$$\text{Ш}(-x, -y) = \text{Ш}(x, y)$$  \hspace{1cm} (6)

$$\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \text{Ш}(x, y) dx dy = 1 \text{ because its periodic with unit period}$$  \hspace{1cm} (7)
Multiplication of a function by $III(x, y)$ effectively samples it at unit intervals.

$$III(x, y)f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(n, m)\delta(x - n, y - m)$$

**Figure:** Sampling a function
\[ \mathbb{I}(x, y) \otimes f(x, y) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f(x - n, y - m) \]

**Seaperability:** \( \delta(x, y) = \delta(x)\delta(y) \), similarly, \( \mathbb{I}(x, y) = \mathbb{I}(x)\mathbb{I}(y) \).

**Derivative of the Impulse function:** We use the limit definition of the impulse function and take the derivative of the function whose limit defines the impulse function. Then take the limit.

**Derivative Sifting Property:**

\[ \delta'(x, y) \otimes f(x, y) = \int\int_{-\infty}^{\infty} \delta'(x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta = f'(x, y) \]

Similarly, \( \delta''(x, y) \otimes f(x, y) = f''(x, y) \)
**LSI Performs Convolution**

- **Proof:** Using the sifting property of the impulse function,
  \[
  f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta \tag{9}
  \]

- Decompose \(f(x, y)\) into elementary functions and then we can determine the overall output \(g(x, y)\) by summing the shifted scaled impulses. Why? (because we can use the fact that the system is linear)

- Response of the LSI system to \(\alpha \delta(x - \xi, y - \eta)\) is \(\alpha h(x - \xi, y - \eta)\) as the system is shift invariant. But \(\alpha = f(\xi, \eta)\).

- Hence,
  \[
  g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta \tag{10}
  \]
  Which is the same as \(g(x, y) = f(x, y) \otimes h(x, y)\). QED.
Properties of Convolution

- Convolution is commutative: $f \otimes g = g \otimes f$.
- Convolution is associative: $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.
- Can cascade systems etc.
- Reading Assignment: read section 6.3 on MTFs.