ONE DIMENSION

In one dimension we define the Fourier transform $F(s)$ of a given function $f(x)$ by

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi sx} \, dx.$$ 

We may think of the right-hand side as specifying an operation of analysis as follows. Multiply the given function $f(x)$ by the factor $\exp(-i2\pi sx)$, a function of $x$ containing a parameter $s$ which is to remain fixed at some constant value for the time being; then integrate the product over all $x$. The result is no longer a function of $x$ but does depend on the chosen value of the parameter $s$. The $F$ value so obtained may now be supplemented by other $F$ values corresponding to other choices of $s$ by repeating the multiplication and integration. The computation performed in this way may be thought of as analysis of $f(x)$ into exponential components. The underlying idea is that exponential components of $f(x)$ not having the chosen $s$ value will contribute products that integrate to nothing, whereas the component at the chosen value of $s$ will respond to the analysis because the integral of its product can be nonzero.

The inverse relationship

$$f(x) = \int_{-\infty}^{\infty} F(s)e^{2\pi sx} \, ds$$

can be thought of as specifying an operation of synthesis, in which exponential functions of $x$ with different $s$ values, each having its appropriate amplitude $F(s) \, ds$, are summed. The function $f(x)$ is thus synthesized from exponentials of all frequencies $s$. Of course, we are talking about exponential functions of imaginary argument, a convenient way of handling sinusoids and cosinusoids simultaneously.

THE FOURIER COMPONENT IN TWO DIMENSIONS

In two dimensions, similar viewpoints are obtained. The two-dimensional Fourier transform $F(u, v)$ of the two-dimensional function $f(x, y)$ is defined by

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-i2\pi(ux + vy)} \, dx \, dy.$$ 

We may view this as an operation of analysis and ask what are we analyzing $f(x, y)$ into. The answer is functions of the form $\exp[-i2\pi(ux + vy)]$ with various amplitudes depending on the choice of $u$ and $v$. The exponential can be decomposed into two terms

$$\cos[2\pi(ux + vy)] \quad \mbox{and} \quad \sin[2\pi(ux + vy)]$$

or into four terms

$$\cos 2\pi ux \cos 2\pi vy, \quad \sin 2\pi ux \sin 2\pi vy, \quad \sin 2\pi ux \cos 2\pi vy, \quad \mbox{and} \quad \cos 2\pi ux \sin 2\pi vy.$$ 

We see that functions $f(x, y)$ that are symmetrical about both the $x$- and $y$-axes may be analyzed into functions of the form $\cos 2\pi ux \cos 2\pi vy$ and that other combinations of
Three or More Dimensions

Clearly in three dimensions the basic formulas generalize to

\[ F(u, v, w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)e^{-i2\pi(ux + vy + wz)} \, dx \, dy \, dz \]

\[ f(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v, w)e^{i2\pi(ux + vy + wz)} \, du \, dv \, dw \]

and similarly in four dimensions or more. Fourier analysis in three dimensions is best known historically from x-ray diffraction analysis of crystals, but applications to elasticity, magnetism and many other fields are numerous for the fundamental reason that space is three dimensional. Four-dimensional transforms arise in plasma dynamics and other branches of fluid dynamics where time variation is inherent. Many authors find it convenient to use compact vector notation that eliminates the multiple integration signs, even if only one integral is saved. In this view \( f(x, y, z) \) becomes \( f(x) \), where \( x \) is a vector whose components are \( x, y, \) and \( z \) and the transform is \( F(u) \).

Vector Form of Transform

The point \( (x, y) \) on the plane, or \( (x, y, z) \) in space, or \( (x, y, z, t) \) in the space-time continuum, can be expressed compactly by a multidimensional vector \( x \) having the components stated. For the \( (u, v) \)-plane, \( (u, v, w) \)-space or the \( (u, v, w, f) \) spatiotemporal frequency domain, a multidimensional vector \( u \) can be similarly defined. The Fourier transform relations can then be succinctly stated, for any dimensionality, using vector dot-product notation \( u \cdot x \) to condense any of the expressions \( ux + vy \), \( ux + vy + wz \), or \( ux + vy + wz + ft \). Thus, for any number of dimensions one can write the one pair of relations

\[ F(u) = \int f(x)e^{-i2\pi u \cdot x} \, dx \]

\[ f(x) = \int F(u)e^{i2\pi u \cdot x} \, du. \]
The differentials \( dx \) and \( du \), which are not vectors, stand for the area elements \( dx \, dy \) and \( du \, dv \), volume elements \( dx \, dy \, dz \) and \( du \, dv \, dw \), and so on. The scalar area element \( dx \) defined as \( dx \, dy \), is to be distinguished from the vector differential \( d \mathbf{x} \) whose components are \( (du, dv) \). The spaces \( X \) and \( U \) are the whole infinite domains of \( x \) and \( u \); consequently one integral sign stands for \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \) or \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \) or more.

This condensed notation is very handy in contexts such as papers where all the integrals are triple integrals or where extended algebra is needed. However, in a book about two dimensions, where single and double integrals are both frequently encountered, it is better for explanatory purposes to distinguish between one-dimensional and two-dimensional integrals and to write them out in full.

**THE CORRUGATION VIEWPOINT**

There is another way of viewing the Fourier component in two dimensions. Let us define a "corrugation" (Fig. 4-2) as a surface generated as the locus of a level straight line that passes through a sinusoid perpendicular to the plane containing that sinusoid. Then the two-dimensional Fourier synthesis can also be described in terms of the superposition of corrugations having all possible wavelengths \( q^{-1} \) and all possible orientations \( \phi \), with appropriate amplitudes. The interrelation is

\[
q = (u^2 + v^2)^{1/2}, \quad \tan \phi = v/u.
\]

where \( q \) is the spatial frequency of a corrugation (the number of waves per unit length in the direction normal to the wave crests) and \( \phi \) is the angle between that direction and the \( x \)-axis. Alternatively we may write

\[
u = q \cos \phi, \quad v = q \sin \phi.
\]

The corrugation viewpoint provides our clearest way of understanding and remembering the basic significance of the variables \( u \) and \( v \). Imagine a sandy desert where the sand dunes have a sinusoidal cross section and all run parallel to each other. Then if you ride a camel (Fig. 4-3) from west to east, \( u \) is the number of crests per unit horizontal distance traveled. The crest-to-crest distance going west to east is \( u^{-1} \). Likewise, if you ride north, the number of crests per meter is \( v \) (assuming that distance is measured in meters), and \( v^{-1} \) is the number of meters per crest. The true crest-to-crest distance measured perpendicular to the direction of the crest lines is shorter than either \( u^{-1} \) or \( v^{-1} \), in general, and, of course, it is the hardest direction in which to ride. The spatial frequency in that direction is greater than in any other direction and may be calculated by vectorial addition of the spatial frequency components (Fig. 4-4) along any pair of orthogonal axes.

Since any surface can be analyzed into corrugations of appropriate amplitude, direction, and spatial frequency, it follows that a quilted component can be so analyzed. We find that it is just the sum of two corrugations that are equally inclined to the coordinate axes. To prove this we reflect the corrugation \( \cos(2\pi(ux + vy)) \) in the \( y \)-axis to obtain \( \cos(2\pi(ux - vy)) \) and add the two together. Then
four null lines bounding the central maximum of their combination. By fixing attention on a point on one of these null lines, one can see how the contributions from the two corrugations do indeed cancel along noninclined loci.

Here is a physical interpretation of the intimate relation between the quilt and corrugation patterns. Suppose the \( x \)-axis is a perfectly reflecting barrier to waves that are incident from the NNE and, after reflection, will be traveling toward the NNW. The previous figure catches these waves at the moment when a crest of the incident wave train is at the origin, and the same is true for the reflected wave. Consequently, the superposition of the two waves has its maximum at the origin, and the full pattern representing the interference of the two oblique wave trains is the quilt pattern. Of course, the lower half of the figure is to be ignored. The two traveling waves interfere to set up a standing wave in the \( y \)-direction. For example, there will be nodal lines of zero disturbance running parallel to the \( x \)-axis, the first one being at a distance \( \frac{1}{2} u^{-1} \) from the reflecting barrier on the \( x \)-axis. The standing wave is a consequence of the fact that energy flow to the south is blocked; on the other hand, there is no barrier to energy flow to the west, therefore the whole quilt pattern is to be regarded as moving to the west. Only at the moment previously frozen in time would the positive maximum be over the origin. After the time taken for the incident wave train to deliver its next crest to the origin, the quilt pattern will have moved west by a distance \( u^{-1} \). Since this distance, the east-west crest-to-crest distance, is greater than the wavelength \( q^{-1} \), we see that the quilt pattern moves faster than the incident waves. This phenomenon can be confirmed occasionally on beaches where the waves are obliquely incident (usually they are not), and the situation also arises in other cases—for example, in glass, where a ray of light is reflected when it impinges on the glass-to-air boundary. The electromagnetic disturbance then propagates along the glass boundary faster than light. The same happens with microwaves in a waveguide for the same reason.

### EXAMPLES OF TRANSFORM PAIRS

A stock-in-trade of two-dimensional Fourier transform pairs is useful for illustrating the analysis and synthesis discussed above and also for illustrating the theorems that follow.

**Impulse**

As a first example consider the pair

\[
\delta(x, y) \rightarrow 1.
\]

which is illustrated in Fig. 4-6. Thus, a centrally situated unit two-dimensional impulse transforms into a function that is equal to unity, independent of \( u \) and \( v \). To establish this we use the sifting property

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) f(x, y) \, dx \, dy = f(0, 0),
\]

from which it follows that
The Two-Dimensional Fourier Transform  Chap. 4

Figure 4.6 The unit two-dimensional impulse and its two-dimensional Fourier transform. The small ticks in this and other illustrations mark unit value along the axes.

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) e^{-2\pi (ax + by)} \, dx \, dy = 1. \]

The inverse transform would be

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi (ax + by)} \, du \, dv = \delta(x, y). \]

To prove this relationship we could, if we wished, take the position that the direct transform has already been established and that, since the impulse notation must conform with ordinary notation, the inverse transform holds by necessity. On the other hand, one could ask for more direct insight. In that case the procedure is to consider a sequence of functions, such as the sequence generated by the expression \( \exp[-\pi \tau^2(x^2 + y^2)] \) as \( \tau \to 0 \), which has the property of approaching unity. Each member of the sequence has a calculable transform. We then look at the sequence of transforms to see whether it is a suitable defining sequence for \( \delta(u, v) \). We will be able to do this below.

As to physical interpretation one could say that sound escaping through a pinhole in a rigid wall spreads uniformly in all directions. This statement will be made rigorous later; meanwhile, it should be noted that further interpretation is needed, because the directions available to the sound reach only to \( 90^\circ \) from the normal.

The inverse transform can be illustrated in the same field of acoustics, where it means that a wavefront of constant amplitude and phase over some plane radiates only in the single direction normal to the plane.

This sort of physical interpretation, whether in terms of acoustics, light, electromagnetic waves, or other waves, is of great assistance in thinking about the mathematical material.

Impulse pair

Taking two half-strength impulses at \((a, 0)\) and \((-a, 0)\) and integrating by means of the sifting property as before, we obtain (Fig. 4.7)

\[ 0.5 \delta(x + a, y) + 0.5 \delta(x - a, y) \supset \cos 2\pi au. \]

Although the direct transform

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [0.5 \delta(x + a, y) + 0.5 \delta(x - a, y)] e^{-2\pi (ax + by)} \, dx \, dy \]

is very easy to evaluate by splitting into two integrals and using the sifting property, the inverse transform

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos 2\pi au \, e^{2\pi (ax + by)} \, du \, dv \]

is, as in the previous example, much more complicated.

One learns that it very often happens that the forward transform and the inverse transform are of unequal difficulty and that it pays to look at both ways of proceeding before forging ahead with what may prove to be the hard way.

As to physical interpretation, here we are dealing with the cosinusoidal interference pattern produced when waves escape through two pinholes, or from two point sources, with the same phase and amplitude at each source point. As before, the inverse transform also has an interpretation and one that is different in character from the interpretation of the direct transform. If we could impose a cosinusoidal amplitude variation, of spatial period \( a^{-1} \), over a plane wavefront, then, according to the transform pair, radiation would be launched in just two directions, equally but oppositely inclined to the normal to the plane. The angle of launch, which is fixed by the spatial period \( a^{-1} \), can easily be deduced.

Gaussian hump

A two-dimensional Gaussian function \( \exp[-\pi (x^2 + y^2)] \), which may also be written \( e^{-a^2} \) in terms of the polar coordinate \( r \), arises in innumerable connections. It is its own Fourier transform (Fig. 4.8),

\[ e^{-\pi r^2} \supset e^{-\pi q^2}. \]

In this statement we use \( q \) as the radial polar coordinate in the \((u, v)\)-plane; thus

\[ r^2 = x^2 + y^2 \quad \text{and} \quad q^2 = u^2 + v^2. \]

The coefficient \( \pi \) is included in the position shown partly to have the convenience of symmetry, which makes it easier to remember this pair, and partly because \( \exp(-\pi x^2) \) has unit area and \( \exp(-\pi r^2) \) has unit volume.

To derive the result requires evaluating the integral in a conventional manner or, as is customary with standard integrals, looking it up in a table of integrals.
Rectangular

A unit rectangle function \( \text{rect}(x, y) = \text{rect} x \text{ rect} y \) transforms into the product of two sinc functions as follows (Fig. 4-9):

\[
\text{rect}(x, y) \supset \text{sinc} u \text{ sinc} v.
\]

Since in one dimension \( \text{rect}(t) \) transforms into \( \text{sinc}(t) \), one might suspect that there is a very simple direct derivation of this example. It will be given below. Radiation from a square aperture has a directional dependence connected with \( \text{sinc} u \text{ sinc} v \). As we are often concerned with power rather than with amplitude in radiation problems, it will often be \( \text{sinc}^2 u \text{ sinc}^2 v \) that is encountered. Apart from any importance that rectangular structures may have in the world of engineering, the two-dimensional rectangle function, which in speech is more conveniently called \( \text{rect} x \text{ rect} y \), arises in various other ways. For example, it is a two-dimensional gate function which, by multiplication, selects out a portion of a field for retention while putting the surrounding function values to zero.

Pillbox

Corresponding to the square aperture is the equally important circular aperture. Interpreted in two dimensions, the unit rectangle function of radius \( \text{rect} r \) or \( \Pi(r) \), represents a function that is equal to unity over a central circle of unit diameter and zero elsewhere. Just as the Fourier transform of a rectangle function in one dimension is a sinc function, so the two-dimensional transform of \( \text{rect} r \) is a jinc function. Naturally, a function as fundamental as the diffraction pattern of a circular aperture is itself intrinsically simple in nature. However, the understanding of the jinc function will be deferred for the moment. Here we content ourselves with introducing the formal definition \( \text{jinc} q \equiv J_1(\pi q)/2q \), where \( J_1 \) is the Bessel function of the first kind of order unity. Then (Fig. 4-10)

\[
\text{rect} r \supset \text{jinc} q.
\]

Gaussian ridge

Consider \( \exp(-\pi x^2) \) to show what happens when the function \( f(x, y) \) is independent of \( y \), i.e., when its representation is a cylinder (in the sense of a surface generated by a straight line kept parallel to \( y \)-axis). The surface of revolution so generated is a cone. Consider another example in which \( f(x, y) \) is independent of \( x \). In this case the surface of revolution is a cylinder.}

Examples of Transform Pairs

Figure 4-9 Unit two-dimensional rectangle function \( \text{rect}(x, y) = \text{rect} x \text{ rect} y \) transforms into a function \( \text{sinc} u \text{ sinc} v \) that is suggested by its two principal cross-sections.

Figure 4-10 Introducing the jinc function, the circular analogue of the sinc function and two-dimensional Fourier transform of the unit pillbox function \( \text{rect} r \). Heights are exaggerated by a factor 2.

Note that this is a case where the sign \( \supset \) is being used for "has two-dimensional Fourier transform" because a possibility of confusion might exist with the symbol \( \supset \) "has Fourier transform." To avoid such risk, where it exists, one can write \( \supset \).

Line impulse

We next have a pair that could be generated from the preceding one by considering a sequence of Gaussian cylinders \( \tau^{-1} \exp(-\pi x^2/\tau^2) \) which, as \( \tau \to 0 \), would be a suitable defining sequence for \( \delta(x) \). If we regard \( \delta(x) \) as a function of two dimensions, but independent of \( y \), we have a line impulse running along the \( y \)-axis. The transforms pass through a sequence of bladed of unit central height whose extent increases as \( \tau^{-1} \). In the limit, the two-dimensional
Two-dimensional signum function
The function sign(x) equals +1 for positive x, -1 for negative x, and zero between. Introduce a two-dimensional generalization \( \text{sgn}(x, y) = x \text{sgn}(y) \), which equals +1 in the first and third quadrants, -1 in the second and fourth, and zero on the axes. Alternatively, \( \text{sgn}(x, y) = \text{sgn}(|x|) \). We pronounce sign(x) [sign] (ks), (using International Phonetic Association symbols), in recognition of the original Latin; this pronunciation avoids a homophone with sin x.

\[
\text{sgn}(x, y) \equiv \frac{1}{\pi x y}
\]

Dipole
The following pair appears in the discussion of infinitesimal dipoles.

\[
\sin 2\theta \equiv \delta(x, y) \sin 2\phi
\]

Gaussian with angular variation
The following self-reciprocal pair has significance as an eigenfunction of the Radon transform as discussed under tomography.

\[
e^{-\pi x^2 \sin 2\theta} \equiv e^{-\pi y^2 \sin 2\phi}
\]

Bed-of-nails or shah function
Just as in one dimension the shah function is its own transform,

\[
\text{III}(x) \equiv \text{III}(x)
\]

so also in two dimensions (Fig. 4-13)

\[
\text{III}(x, y) \equiv \text{III}(x, y)
\]

Grid
Regarded as a function of two variables the shah function

\[
\text{III}(x, y) = \sum_{m=-\infty}^{\infty} \delta(x - n)
\]

describes a set of vertical unit-strength line impulses uniformly spaced at unit interval. Likewise \( \text{III}(y) \) describes a horizontal grid. The product of \( \text{III}(y) \) with \( f(x, y) \) retains values of \( f(x, y) \) along the lines \( x = \text{integer} \) but abandons values in between. Thus \( \text{III}(y) f(x, y) \) contains information that is preserved in a horizontal raster scan of the function \( f(x, y) \). Likewise \( \text{III}(x) f(x, y) \) would represent the information in a vertical raster scan, which is not seen very often because of the convention that has been adopted with television. The two-dimensional transform of \( \text{III}(x) \) is a uniform row of unit impulses \( \sum_{m=-\infty}^{\infty} \delta(u - m, v) \) which can also be written briefly as \( \text{III}(u) \delta(v) \). The transform pair is

\[
\text{III}(x, y) \equiv \text{III}(u) \delta(v).
\]

Conversely, \( \text{III}(y) \equiv \text{III}(u) \delta(v) \), a row of spikes on the v-axis, as illustrated in Fig. 4-14.
REMS FOR TWO-DIMENSIONAL FOURIER TRANSFORMS

Similarity Theorem

All the theorems for the one-dimensional Fourier transform go over into two dimensions. The similarity theorem, which tells us what happens when the abscissa expands or contracts, generalizes in two dimensions to tell us the effect of expansion or contraction of the picture plane (Fig. 4-15). An extra element enters, inasmuch as the expansion need not be uniform but may be unequal along the two axes. The theorem does not tell us what happens if the picture plane is stretched diagonally. Furthermore, it is not possible to combine stretching and compression along the coordinate axes to simulate stretching in the 45° direction. However, stretching in an arbitrary direction is easily handled with the aid of the rotation theorem introduced below.

Similarity Theorem in One Dimension

If \( f(x) \triangleright F(s) \)
then \( f(ax) \triangleright |a|^{-1} F(s/a) \).

Similarity Theorem in Two Dimensions

If \( f(x, y) \triangleright F(u, v) \)
then \( f(ax, by) \triangleright |ab|^{-1} F(u/a, v/b) \).

Shift Theorem

When a one-dimensional function is translated along the axis of abscissas, or, what is equivalent, the origin is shifted, no change occurs in the amplitude of any Fourier component, but the phases will be changed. A given amount of shift will amount to more and more phase change as components of higher frequency are considered. For example, consider a function of time that is delayed by an amount \( \tau \), and think of the effect on the Fourier component \( A \cos 2\pi f t \). The period of this component is \( f^{-1} \), and, as a shift of one period would introduce a phase change \( 2\pi \), we see that a delay \( \tau \) will change the phase by \( -2\pi f \tau \). We see that the phase change of the component not only depends on its frequency \( f \) but is proportional to \( f \) (and is also, of course, proportional to the delay \( \tau \)). A shift in the function domain thus introduces a linear phase gradient in the transform domain.

Shift Theorem in One Dimension

If \( f(x) \triangleright F(s) \)
then \( f(x-a) \triangleright e^{-2\pi i a F(s)} \).

By phase gradient we mean that the phase shift itself is not constant, but that a constant rate of change of phase is introduced along the axis of abscissas of the transform. The phase change at the origin \( s = 0 \) is zero. In two dimensions the origin can be independently translated along either the \( x \)- or \( y \)-axes, which leads to a simple generalization.
Shift Theorem in Two Dimensions

If \( f(x, y) \ Subscriber to a real function, it becomes complex; as a result, the inverse shift theorem might not be expected to arise very often, but in fact it often helps to think in terms of shifting things around on the \((u, v)\)-plane.

Again a suggestion arises of theorems with no one-dimensional counterpart. In this case we can imagine a displacement that comprises not only a two-dimensional translation but also a rotation. While displacement in one-dimension can be described by just one parameter, the most general displacement in two dimensions, without change of form, requires three parameters or can be said to possess three degrees of freedom.

Rotation Theorem

It is physically obvious that rotation of an antenna about its beam axis will result in an equal rotation of the radiation pattern of the antenna, and the same can be said for the diffraction pattern of an optical source. We know that a Fourier transform underlies the relationship in these cases, and so we expect the following theorem.

Rotation Theorem

If \( f(x, y) \subseteq F(u, v) \)

then \( f(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) \subseteq F(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta). \)

It is not intuitively obvious from the mathematical definition that this theorem follows, and in particular some moments of very attentive mathematical thought are required to assure oneself that clockwise rotation of \( f(x, y) \) does not result in counterclockwise rotation of \( F(u, v) \). This theorem is an example of the kind of power that results from possessing more than one way of interpreting Fourier transforms.

Shear Theorems

The similarity theorem, shift theorem, and rotation theorem are special cases where one distorts a function by shifting points on the \((x, y)\)-plane in accordance with an affine transformation

\[ x' = ax + by + c, \quad y' = dx + ey + f. \]

Pure shear is another special case. For example,

\[ x' = x + by, \quad y' = y \]

describes horizontal shear distortion that would take each square on a sheet of graph paper into an equilateral parallelogram, or rhombus, of the same height and area but sloping sides. A general function \( f(x, y) \) is converted into a different function \( f(x + by, y) \). The shear theorem tells us that there is a corresponding distortion of the \((u, v)\)-plane which alters the original transform \( F(u, v) \), as follows.
Simple Shear Theorem

If \(f(x, y) \supseteq F(u, v)\)
then \(f(x + by, y) \supseteq F(u, v - bu)\).

Thus the corresponding distortion in the \((u, v)\)-plane is also pure shear but in the perpendicular direction. As is known from the theory of elasticity, a state of shear strain is equivalent to a combination of pure compression and extension in diagonal directions. One way of proving the shear theorem is to apply the similarity theorem in two perpendicular diagonal directions in turn.

Vertical shear is described by \(x' = x, y' = dx + y\), and in this case the transform suffers horizontal shear to become \(F(u - dv, v)\).

\[\text{Compound Shear Theorem}\]

If \(f(x, y) \supseteq F(u, v)\)
then \(f(x + by, dx + y) \supseteq \frac{1}{|1 - bd|} F\left(\frac{u - dv}{1 - bd}, \frac{-bu + v}{1 - bd}\right)\).

Looking at \((x, y)\) as a vector \(x\) and \((x', y')\) as a vector \(x'\), the compound-shear coordinate transformation can be written concisely as

\[x' = \begin{bmatrix} 1 & b \\ d & 1 \end{bmatrix} x.\]

The matrix operator for horizontal shear is \(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\) or \(\begin{bmatrix} 1 & \tan \beta \\ 0 & 1 \end{bmatrix}\) and for vertical shear \(\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}\) or \(\begin{bmatrix} 1 & 0 \\ \tan \delta & 1 \end{bmatrix}\), where \(\beta\) and \(\delta\) are angles noted on Fig. 4-17. Horizontal shear followed by vertical shear is expressed by

\[\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ d & 1 + bd \end{bmatrix},\]

while vertical shear followed by horizontal shear is expressed by

\[\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} = \begin{bmatrix} 1 + bd & b \\ d & 1 \end{bmatrix}.\]

Thus simple shear operations carried out in succession produce different outcomes, depending on the order in which they are applied. In matrix terminology, the matrix factors do not commute. In neither case is the outcome the same as for compound shear as defined. Compound shear results in the same inclination angles \(\beta\) and \(\delta\) as for simple shear. Figure 4-17 shows that \(\delta\) is retained in case (d) while a new angle \(\zeta\) is introduced. In case (e) \(\beta\) is retained and a new angle \(\eta\) comes in. The angles \(\zeta\) and \(\eta\) are given by \(\tan \zeta = b/(1 + bd)\) and \(\tan \eta = d/(1 + bd)\). Since shear does not change the area of a figure, all the deformed outlines in Fig. 4-17 retain unit area. The area magnification is equal to \(|ae - bd|\).

\[\text{Table 4-1 Reference Table for Shear}\]

<table>
<thead>
<tr>
<th>Case</th>
<th>(B)</th>
<th>(C)</th>
<th>(D)</th>
<th>(m_{AB})</th>
<th>(m_{AC})</th>
<th>(m_{AD})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
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<td>(1 + b, 1)</td>
<td>(b, 1)</td>
<td>0</td>
<td>1/(1 + b)</td>
<td>1/b</td>
</tr>
<tr>
<td>(b)</td>
<td>(1, d)</td>
<td>(1 + b, d)</td>
<td>(b, d)</td>
<td>d</td>
<td>1</td>
<td>\infty</td>
</tr>
<tr>
<td>(c)</td>
<td>(1, d)</td>
<td>(1 + b, 1 + d)</td>
<td>(b, 1)</td>
<td>d/(1 + b)</td>
<td>1/b</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>(1, d)</td>
<td>(1 + b, 1 + bd + d)</td>
<td>(b, 1 + bd)</td>
<td>d</td>
<td>(1 + bd + d)/(1 + b)</td>
<td>(1 + bd)y/b</td>
</tr>
<tr>
<td>(e)</td>
<td>(1 + bd, d)</td>
<td>(1 + b + bd, 1 + d)</td>
<td>(b, 1)</td>
<td>d/(1 + bd)</td>
<td>(1 + bd)/(1 + b)</td>
<td>1/b</td>
</tr>
</tbody>
</table>

A unit square \(ABCD\) with \(A\) at the origin, \(B\) at \((1, 0)\), \(C\) at \((1, 1)\), and \(D\) at \((0, 1)\), after deformation under the five cases of Fig. 4-17, finds its vertices in the locations listed in the accompanying table. The point \(A\) remains at \((0, 0)\). The slope of the side \(AB\) is listed under \(m_{AB}\) and similarly for the diagonal \(AC\) and initially vertical side \(AD\).

Other natural parameters are the expansion or compression of the diagonals, which are easily calculable but, if conversely the extension of the diagonal is specified, it is awkward to extract the parameters \(a, b, c, d\). Bearing in mind that CCW rotation through \(\theta\) is produced by the operator \(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}\), while expansion by a factor \(M\) in the \(x\)-direction is produced by \(\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}\), any desired expansion in a specified direction can be reached by a combination of rotation and expansion and expressed in the form \(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\) by a sequence of matrix multiplications.

**Affine Theorem**

This is a little-known theorem that incorporates several of the foregoing theorems (similarity, shift, compound, shear) as special cases. It is particularly useful in its general form in image processing, where sequences of affine transformations are applied. Derivations have
not been given for the simpler theorems, but the derivation of this theorem is instructive (Bracewell et al., 1993). The theorem which we shall derive is:

**Affine Theorem**

If \( f(x, y) \) has 2-D FT \( F(u, v) \), then \( g(x, y) = f(ax + by + c, dx + ey + f) \) has 2-D FT

\[
G(u, v) = \frac{1}{|\Delta|} \exp \left\{ \frac{i2\pi}{\Delta} \left[ \begin{array}{c} (ec - bf)u + (af - cd)v \\ \end{array} \right] \right\} F \left( \frac{eu - dv}{\Delta}, \frac{-bu + av}{\Delta} \right),
\]

where the determinant \( \Delta \) is given by

\[
\Delta = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd.
\]

To derive this result, express the affine coordinate transformation

\[
x' = ax + by + c, \quad y' = dx + ey + f
\]

in the matrix notation:

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}
\]

and note the Jacobian relation \( dx'dy' = |\Delta| dx dy \). If \( \Delta \neq 0 \), invert the transformation to get

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} x' - c \\ y' - f \end{bmatrix}.
\]

In the phase exponent \(-i2\pi(u'x + v'y)\) that occurs in the definition of the two-dimensional Fourier component note that

\[
u x + v y = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1} \begin{bmatrix} x' - c \\ y' - f \end{bmatrix}.
\]

\[
= \frac{1}{\Delta} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} e & -b \\ -d & a \end{bmatrix} \begin{bmatrix} x' - c \\ y' - f \end{bmatrix}.
\]

Hence

\[
G(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(ax + by + c, dx + ey + f) e^{-i2\pi(u'x + v'y)} dx dy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') e^{-i2\pi \left( (ue - dv)x' + (-bu + av)y' \right) / \Delta} \frac{1}{\Delta} \left( (ec - bf)u + (af - cd)v \right) dx' dy' / |\Delta|.
\]

\[
= \frac{1}{|\Delta|} \exp \left\{ \frac{i2\pi}{\Delta} \left[ \begin{array}{c} (ec - bf)u + (af - cd)v \\ \end{array} \right] \right\} F \left( \frac{eu - dv}{\Delta}, \frac{-bu + av}{\Delta} \right).
\]

This completes the derivation. The expression can be condensed by referring to affine transform plane coordinates defined by \( u' = (eu - dv) / \Delta \) and \( v' = (-bu + av) / \Delta \). Then

\[
G(u, v) = \frac{1}{|\Delta|} e^{i2\pi(c'x' + f'y')} F(u', v').
\]

The inverse transformation to \((u, v)\)-coordinates is

\[
\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & d \\ b & e \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}.
\]

One does not begin with much intuitive feel for the affine coefficients: taken on their own, \( b \) and \( d \) express \( x \)-shear and \( y \)-shear, \( a \) and \( e \) are linear magnifications, while \( c \) and \( f \) are displacements. It would make sense to introduce mnemonic value by writing

\[
x' = M_x x + \sigma_x y + x_0
\]

\[
y' = M_y y + \sigma_y y + y_0.
\]

In this notation some expressions seem to be more intelligible. For example, the area magnification \(|ae - bd|\) becomes \(|M_x M_y - \sigma_x \sigma_y|\).

**Rayleigh's Theorem in Two Dimensions**

Although Rayleigh's theorem, first published in 1889, referred to time-dependent waveforms and their spectra, it is convenient to generalize the name to include the corresponding theorem in two dimensions. The theorem is as follows:

**Rayleigh's Theorem in Two Dimensions**

If \( f(x, y) \to F(u, v) \)

then

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f(x, y) \right|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| F(u, v) \right|^2 du dv.
\]

In physical situations the relation often represents the equality of two power flows. For example, the left-hand side might represent the power flowing through an aperture as integrated element by element over the aperture, and the right-hand side might be the same power integrated direction by direction over the diffracted beam of the radiated field.

There is a more general theorem, which arises less often. It would be encountered in connection with radiation from an aperture, if the power were to be expressed as the...
The product of an electric with a magnetic field rather than as proportional to the squared magnitude of just one field component. The more general theorem is
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)g^*(x, y)\, dx\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)G^*(u, v)\, du\, dv. \]

A derivation of Rayleigh's theorem can be based directly on the autocorrelation theorem, which will be introduced later.

Parseval's Theorem in Two Dimensions

Parseval's theorem, like Rayleigh's, can often be interpreted as a statement of equality between two energies or powers viewed in different ways. For example, in a loss-free electric transmission-line system the stored energy can be expressed as a spatial integral of the squared modulus of the electric field, or it can be expressed as the sum of the energies in the natural modes. In a mechanical system or in an acoustical resonator the sum of the energies in all the modes of vibration is equal to a spatial integral of squared stress or strain or of sound intensity.

If there is a periodic waveform \( f(t) \), then \( \int f(t)^2 \, dt \) cannot be integrated over infinity, but it can be integrated over one period; theorems involving \( \int_{-\infty}^{\infty} f(t)^2 \, dt \) do not apply to periodic functions. Correspondingly, if \( f \) is not periodic, then \( \int |F|^2 \, dt \) cannot be evaluated if \( F \) is impulsive, as it will be if \( f \) is periodic. Parseval's theorem deals with such situations. Let \( f(x, y) \) be periodic in both \( x \) and \( y \) with unit period. The theorem relates the integral of \( |f(x, y)|^2 \) over unit cell to the coefficients of the impulses in the \((u, v)\)-plane defined by
\[ F(u, v) = \sum \sum a_{mn} [\delta(m - u, v - n)]. \]

Parseval's Theorem in Two Dimensions

If \( f(x + 1, y + 1) = f(x, y) \) for all \( x \) and \( y \)
then \[ \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \left| f(x, y) \right|^2 \, dx\, dy = \Sigma \Sigma \left( \frac{2}{a^2} \right) a_{mn}, \]
where \( a_{mn} \) is the strength of the impulse at \( u = m, v = n \).

Marc Antoine Parseval (1755–1836) published his result in 1799 (before Fourier began working on heat diffusion) and also evaluated the spin integral of the cosine function to obtain the series expansion of the zero-order Bessel function.

Derivative Theorem

It is sufficient to discuss differentiation with respect to \( x \), for which the theorem is as follows.

Derivative Theorem

If \( f(x, y) \) is differentiable with respect to \( x \), then the derivative theorem follows:
\[ \frac{\partial f(x, y)}{\partial x} = \lim_{a \to 0} \frac{\Delta_x f(x, y)}{a}, \]
where \( \Delta_x f(x, y) = f(x + \frac{1}{2}a, y) - f(x - \frac{1}{2}a, y) \).

Theorems for Two-Dimensional Fourier Transforms

If \( f(x, y) \) and \( F(u, v) \)
then \[ \frac{\partial}{\partial x} f(x, y) \Rightarrow i2\pi u F(u, v). \]

We see that all spatial frequency components of \( f(x, y) \) are to be increased (or reduced) in proportion to their \( x \)-component of spatial frequency and moved \( 90^\circ \) in spatial phase. The reason for this is apparent from consideration of a single component \( \sin 2\pi u x \sin 2\pi v y \) whose derivative with respect to \( x \) is
\[ 2\pi u \sin (2\pi u x + \frac{1}{2}\pi) \sin 2\pi v y. \]

The higher the frequency of a given component, the more its amplitude is enhanced in the spectrum of the derivative, because, of course, the maximum slope of a sinusoid of given amplitude is greater in proportion to its frequency.

A list of further results stemming from this theorem follows.
\[ \frac{\partial}{\partial y} f(x, y) \Rightarrow i2\pi v F(u, v) \]
\[ \frac{\partial^2}{\partial x^2} f(x, y) \Rightarrow -4\pi^2 u^2 F(u, v) \]
\[ \frac{\partial^2}{\partial y^2} f(x, y) \Rightarrow -4\pi^2 v^2 F(u, v) \]
\[ \frac{\partial^2}{\partial x\partial y} f(x, y) \Rightarrow -4\pi^2 uv F(u, v) \]
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(x, y) \Rightarrow -4\pi^2 (u^2 + v^2) F(u, v). \]

Difference Theorems

Define the first difference of \( f(x, y) \) in the \( x \)-direction over interval \( a \) by \( \Delta_x f(x, y) = f(x + \frac{1}{2}a, y) - f(x - \frac{1}{2}a, y) \). By the shift theorem, the transform is \( \exp(iauF(u, v)) \). Thus

First Difference Theorem

If \( f(x, y) \),
then \( \Delta_x f(x, y) = f(x + \frac{1}{2}a, y) - f(x - \frac{1}{2}a, y) \Rightarrow 2i \sin \pi au F(u, v). \)

Since, by definition,
\[ \frac{\partial f(x, y)}{\partial x} = \lim_{a \to 0} \frac{\Delta_x f(x, y)}{a}, \]
the derivative theorem follows from the first difference theorem. The second difference \( \Delta_{xx} f(x, y) = \Delta_x [\Delta_x f(x, y)] \) leads to the
Second Difference Theorem

If \( f(x, y) \rightarrow F(u, v) \), then
\[
\Delta_x f(x, y) = f(x + a, y) - 2f(x, y) + f(x - a, y) \rightarrow -4\sin^2\pi xu F(u, v).
\]

Further results are
\[
\begin{align*}
\Delta_x \Delta_y f(x, y) & \rightarrow -4\sin\pi xu \sin\pi yv F(u, v), \\
\Delta_{xx} + \Delta_{yy} f(x, y) & \rightarrow -4[\sin^2\pi xu + \sin^2\pi yv] F(u, v).
\end{align*}
\]

Definite Integral Theorem

Very useful for checking calculations, for determining normalization factors, and for evaluating integrals, this apparently trivial theorem is constantly drawn on.

\begin{align*}
\text{Definite Integral Theorem} & \\
\text{If } f(x, y) & \rightarrow F(u, v), \\
\text{then } & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = F(0, 0).
\end{align*}

First Moment Theorem

Since the operation of taking first moments comes up in a wide variety of circumstances, it is useful to know the corresponding operation in the transform domain. Just as the first moment of \( f(x, y) \) is merely one piece of information about \( f(x, y) \), so also there is a single property of \( F(u, v) \) which reveals the first moment of \( f(x, y) \). That property is the central slope of \( F(u, v) \) in the \( u \)-direction.

\begin{align*}
\text{First Moment Theorem} & \\
\text{If } f(x, y) & \rightarrow F(u, v), \\
\text{then } & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y) \, dx \, dy = \frac{1}{-2\pi i} F_u'(0, 0),
\end{align*}

where \( F_u'(u, v) = (\partial/\partial u)F(u, v) \).

This theorem can be derived from the inverse of the derivative theorem
\[
-i2\pi x f(x, y) \rightarrow (\partial/\partial u)F(u, v),
\]
where the factor \(-i2\pi x\) takes the place of the factor \(i2\pi u\) seen in the normal derivative relation in the transform domain. Now
\[
-i2\pi x f(x, y) \rightarrow -\frac{\partial}{\partial u} F(u, v),
\]

Note that the result given in Theorem 4.6 for \( F(x, y) \rightarrow f(x, y) \) can be differentiated with respect to \( u \) to yield the result given in the theorem.

Theorem for Two-Dimensional Fourier Transforms

Applying the definite integral theorem to the inverse of the derivative theorem, we get
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -i2\pi x f(x, y) \, dx \, dy = (\partial/\partial u)F(u, v) \bigg|_{u=0}.
\]

Second Moment Theorem

Applying the definite integral theorem after taking the second derivative of \( F(u, v) \) with respect to \( u \), we obtain immediately the next theorem.

\begin{align*}
\text{Second Moment Theorem} & \\
\text{If } f(x, y) & \rightarrow F(u, v), \\
\text{then } & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f(x, y) \, dx \, dy = -F''_{uu}(0, 0)/4\pi^2.
\end{align*}

Other results may be added for reference.
\[
\begin{align*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f(x, y) \, dx \, dy = & -F''_{uv}(0, 0)/4\pi^2, \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x y f(x, y) \, dx \, dy = & -F''_{uv}(0, 0)/4\pi^2, \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)^2 f(x, y) \, dx \, dy = & -[F''_{uu}(0, 0) + F''_{vv}(0, 0)]/4\pi^2.
\end{align*}
\]

Equivalent Area Theorem

Just as an equivalent width can be defined in one dimension, so in two dimensions there is an equivalent area \( A_f \) with the property that
\[
\text{(equivalent area)} \times \text{(height of function at origin)} = \text{volume}.
\]

Thus
\[
A_f f(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy.
\]

The transform \( F(u, v) \) has an equivalent area also. Call it \( A_F \).

\begin{align*}
\text{Equivalent Area Theorem} & \\
\text{If } f(x, y) & \rightarrow F(u, v), \\
\text{then } & A_f = 1/A_F.
\end{align*}

Equivalent area is a standard concept in some fields. For example, the concept of beam solid angle \( \Omega \) of an antenna radiation pattern is the equivalent area of the radiation intensity pattern. Antenna directivity or directive gain \( D \) may also be expressed in terms of equivalent area; in fact \( D = 4\pi/\Omega \), but in other fields equivalent area is never employed but in some cases might be useful. Its simple arithmetic properties make it rather useful.
The Two-Dimensional Fourier Transform

Separable Product Theorem

It may happen that a function of \( x \) and \( y \) is separable into a product of two functions, one of which is a function of \( x \) alone and the other of \( y \) alone. In that case there is a very useful relation that enables the two-dimensional Fourier transform to be determined by taking one-dimensional transforms only.

\[
\text{Separable Product Theorem} \quad \text{If } f(x) \supseteq F(u) \quad \text{and} \quad g(x) \supseteq G(v) \\
\text{then } f(x)g(y) \supseteq F(u)G(v).
\]

In the important special case where the function of \( x \) and \( y \) is independent of \( y \), we can think of \( g(y) \) as being unity. Then it follows that

\[
f(x) \supseteq F(u)\delta(v).
\]

THE TWO-DIMENSIONAL HARTLEY TRANSFORM

Given a function \( f(x, y) \), and using the abbreviation \( \cos \theta = \cos \theta + \sin \theta \), the two-dimensional Hartley transform and its inverse are as follows.

\[
H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cos[2\pi (ux + vy)] \, dx \, dy
\]

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v) \cos[2\pi (ux + vy)] \, du \, dv.
\]

The two-dimensional Hartley transform of a real object has the interesting property of being itself real. With the Fourier transform, which is complex and possesses hermitian redundancy, one-half of the transform plane suffices to determine the original object. Information in the Hartley plane is spread over the whole plane without redundancy or symmetry. The antisymmetry that characterizes the Fourier plane is not possessed by the Hartley plane, every point of which counts. In computing, the property of being real valued is a considerable convenience, as it is also is in analogous situations where phase, which is not responded to by optical detectors, has significance.

A fast algorithm for Fourier analysis of images (faster than the FFT) has been discovered and may be useful to those concerned with filtering or other transform domain manipulation of images (Bracewell et al., 1986).

THEOREMS FOR THE HARTLEY TRANSFORM

All the theorems for the Fourier transform have counterparts applying to the Hartley transform. The several theorems that are special cases of the affine theorem (Bracewell, 1994) can be condensed for reference as follows.

Discrete Transforms

Affine Theorem for Hartley Transform

If \( f(x, y) \) has 2-D Hartley transform \( H(u, v) \),

then \( f(ax + by + c, dx + ey + f) \) has 2-D Hartley transform

\[
|\Delta|^{-1} \left[ H(\alpha, \beta) \cos \Theta - H(-\alpha, -\beta) \sin \Theta \right] 
\]

where

\[
\Delta = ae - bd \\
\alpha = (eu - dv)/\Delta \\
\beta = (-bu + av)/\Delta \\
\Theta = 2\pi \Delta^{-1} \left[ (ec - bf)u + (af - cd)v \right].
\]

Other theorems can be extracted directly from the Fourier version as follows:

Conversion Theorem

If \( f(x, y) \) has Fourier transform \( R(u, v) + iI(u, v) \),

then \( f(x, y) \) has Hartley transform \( R(u, v) - I(u, v) \).

DISCRETE TRANSFORMS

The discrete Fourier transform and discrete Hartley transform are well known in one dimension and need only be mentioned here in their two-dimensional form for reference. For the discrete Fourier transform

\[
F(\sigma, \tau) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp \left[ -i \left( \frac{2\pi \sigma x}{M} + \frac{2\pi \tau y}{N} \right) \right]
\]

\[
f(x, y) = \sum_{\sigma=0}^{M-1} \sum_{\tau=0}^{N-1} F(\sigma, \tau) \exp \left[ i \left( \frac{2\pi \sigma x}{M} + \frac{2\pi \tau y}{N} \right) \right]
\]

and for the discrete Hartley transform

\[
H(\sigma, \tau) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cos \left( \frac{2\pi \sigma x}{M} + \frac{2\pi \tau y}{N} \right)
\]

\[
f(x, y) = \sum_{\sigma=0}^{M-1} \sum_{\tau=0}^{N-1} H(\sigma, \tau) \cos \left( \frac{2\pi \sigma x}{M} + \frac{2\pi \tau y}{N} \right)
\]

The equations refer to a data array of size \( M \times N \) with integer indices \( x \) and \( y \) running from 0 to \( M - 1 \) and 0 to \( N - 1 \), respectively, and having the same meaning as on the \((x, y)\)-plane apart from the restriction to integer values. The range restriction, which places the origin at the lower left, is awkward at times when it is more natural to think of a centrally situated origin, but it is firmly fixed by custom. Parts of an image that would be
thought of as residing in the NW quadrant with respect to a central origin appear in the SE of the $M \times N$ matrix, SE goes to NW, and SW goes to NE. At the same time reflections are introduced which are correctly implemented when negative values of $x$ and $y$ (relative to a central origin) are replaced by $M - x$ and $N - y$ in the matrix.

Division by the factor $MN$ results in the transform values $F(0,0)$ and $H(0,0)$ being equal to the spatial dc level of $f(x,y)$, as is customary for the leading term in the Fourier series expansion of a periodic function. This factor is absent from the inverse formula. In computing practice, it is efficient not to introduce the factor $MN$ at all into the fast algorithm but to combine it subsequently with other normalizing factors, calibration factors, or graphics scale factors that are needed when the results come to be displayed.

The symbols $\sigma$ and $\tau$, which have to do with spatial frequency, are not customary and are introduced to draw attention to the fact that they differ in meaning from the true spatial frequencies $\nu$ and $v$. To get the spatial frequency in cycles per unit of $x$ (and the unit of $x$ is by definition unity in the discrete case) we need $\sigma / M$, but that is not all; $\sigma$ has to be no larger than $\frac{1}{2} M$. In the range $\frac{1}{2} M < \sigma < M$, the spatial frequency represented by $\sigma$ is $(\sigma - M) / M$. This is because small negative values of $\nu$ appear as large values of $\sigma$ in the discrete formulation. No spatial frequency greater than $0.5$ thus appears, as is expected for sampling at unit spacing, where the shortest period is 2. For $\tau$ the spatial frequency is $(\tau - N) / N$ or $\tau / N$ according as $\tau$ exceeds $\frac{1}{2} N$ or not.

**SUMMARY**

Different ways of looking at two-dimensional spectral analysis have been presented. One way is to generalize from what one already knows about time-domain waveforms and their spectra; another is to think pictorially in terms of topographic surfaces such as sinusoidal dunes of all wavelengths and orientations or alternatively of the "quilted surfaces" which, in general, have unequal spatial frequencies in the $x$- and $y$-directions. The concept of spatial frequency measured in cycles per unit of distance in the $(x,y)$-plane needs to become as familiar as the concept of temporal frequency measured in cycles per second; a crucial difference is that spatial frequency has two components and may be thought of as a complex number.

A stock-in-trade of transforms has been recorded compactly for reference and will be found to cover the majority of needs in everyday image engineering, especially when extended to all the variations generated by the use of the reference list of theorems. Many examples of transform pairs so generated are found in the problems; facility at recognizing the applicability of some theorem to a known pair is most useful to cultivate. Only brief mention of the significance or use of the material has been introduced at this stage in order to keep the presentation brief and compact for reference purposes. Still, it is not a good idea to skim this chapter; take a deep breath, accept it as a heavy mathematical pill, swallow it, and make what you think will be useful in the future part of you.