Homework 3 Due on Thursday, 03/22

Problem 1 (Rayleigh quotient and generalized eigenvalue problem)

In vision and machine learning (and many other disciplines), one frequently encounters optimization problems of the form
\[ E(x) = \max_{x} \frac{x^t P x}{x^t Q x}, \]
where \( x \) is an \( n \)-dimensional vector and \( P, Q \) are two symmetric \( n \times n \) matrices. This kind of quotients is called Rayleigh quotient.

A. The numerator (and also denominator) \( x^t P x \) is a quadratic function in the components of \( x \). For example, if \( x = [x_1, x_2]^t \), we have \( x^t P x = P_{11} x_1^2 + 2 P_{12} x_1 x_2 + P_{22} x_2^2 \), which is a quadratic function of \( x_1 \) and \( x_2 \).

Show that the gradient of \( x^t P x \) is simply the vector \( 2 P x \). This result will be false if \( P \) is not symmetric.

B. Show that \( E \) is constant under scaling, i.e., \( E(x) = E(sx) \) for any \( s \in \mathbb{R} \).

C. Show that the optimization problem above is equivalent to the following constrained optimization problem,
\[ \max_{x} x^t P x, \quad \text{such that} \quad x^t Q x = 1. \]
Using Lagrange multipliers, show that the critical values of this optimization problem are the generalized eigenvalues \( \lambda \) for the two symmetric matrices \( P \) and \( Q \). \( \lambda \) is a generalized eigenvalue of \( P \) and \( Q \) if there exists a nonzero vector \( x \) such that \( P x = \lambda Q x \).

D. Ellipse Fitting Let \( p_1 = [x_1, y_1]^t, p_2 = [x_2, y_2], \ldots, p_n = [x_n, y_n] \) denote \( n \) points in \( \mathbb{R}^2 \) that lie close to an ellipse. A typical ellipse in \( \mathbb{R}^2 \) has an equation of the form
\[ ax^2 + bxy + cy^2 + dx + ey + f = 0, \]
with \( b^2 - 4ac < 0 \). The ellipse fitting problem is to find an ellipse that best fits the given data points.

Let \( a \) denote the 6-dimensional vector \( a = [a, b, c, d, e, f]^t \). For this problem, we use algebraic distance, and we want to find an ellipse with equation given above such that the sum
\[ \mathcal{E}(a) = \sum_{i=1}^{N} |p_i^t a|^2 = \sum_{i=1}^{N} (ax_i^2 + b x_i y_i + cy_i^2 + dx_i + ey_i + f)^2 \]
is minimized. Show that this problem can be reformulated as a constrained quadratic optimization problem of the form
\[ \min a^t P a, \quad \text{such that} \quad a^t Q a = -1, \]
with \( P \) and \( Q \) symmetric. Show the two matrices \( P, Q \).

E. Describe briefly how you would solve the ellipse fitting problem using algebraic distance.
Problem 2 (Singular Value Decomposition)

Let \( A \) denote an \( n \times m \) matrix. Clearly, \( P = A^t A \) and \( Q = AA^t \) are two symmetric matrices. Therefore, all their eigenvalues are real and their eigenvectors form bases in \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively.

A. Show that \( P \) and \( Q \) are both positive semi-definite, i.e., for any nonzero vector \( y \), \( y^t Py \geq 0 \) and same for \( Q \). Explain why the eigenvalues of these two matrices are nonnegative.

B. Let \( u \) be an eigenvector of \( P \) with eigenvalue \( \lambda \). Show that \( Au \) is an eigenvector of \( Q \) corresponding to the same eigenvalue \( \lambda \). Similarly, if \( v \) is an eigenvector of \( Q \) with eigenvalue \( \mu \), then \( A^t v \) is an eigenvector of \( P \) corresponding to the same eigenvalue \( \mu \).

C. Suppose that \( n \) eigenvalues of \( Q \) are all positive, \( \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \). Let \( v_i, 1 \leq i \leq n \) be the corresponding eigenvectors of \( Q \) with unit length, and denote \( u_i = \frac{A^t v_i}{\|A^t v_i\|} \) the unit vector in the direction of \( A^t v_i \). Show that (use B above) there exist real numbers \( \gamma_i \) for \( 1 \leq i \leq n \) such that

\[
Au_i = \gamma_i v_i.
\]

(This problem looks complicated at first. Be patient. Read the problem carefully and you will see that the answer can be as short as two or three sentences in English. This is what mathematics should be! More about thoughts than formulas.)

D. OK, we have defined two sets of unit vectors \{\( v_1, \cdots, v_n \)\} and \{\( u_1, \cdots, u_n \)\} such that \( Au_i = \gamma_i v_i \). Just to make sure you know what you are doing. What are the dimensions of the vectors \( v_i \) and \( u_i \)? And what is the dot product \( u_i^t u_j \) for \( i \neq j \)? Let’s assume for the moment that \( n \leq m \). This implies that there must exist a unit vector \( w \) that is orthogonal to all \( u_1, \cdots, u_n \). What is \( Aw \)?

E. Finally, let \( U = [v_1 v_2 \cdots v_n] \), \( V = [u_1 u_2 \cdots u_n] \) and \( \Sigma \) the diagonal matrix with diagonal entries \( \gamma_1, \gamma_2, \cdots, \gamma_n \). Show that

\[
A = U \Sigma V^t.
\]

This is the singular decomposition of \( A \), where \( \gamma_i \) are the singular values. What can you say about the columns of \( U \) and \( V \)?

F. You have just proved the existence of singular value decomposition (SVD) for the matrix \( A \), perhaps the most important linear algebraic result for computer vision! Take a few minutes to congratulate yourself and try to see if you can go through all the steps without using paper and pen.

By now, we have encountered several instances of the following least-squares problem: Given a matrix \( A \), find a vector \( v \) such that \( \|Av\|^2 \) is minimized with the constraint that \( \|v\|^2 = 1 \) (\( v \) is a unit vector). The standard solution is to compute SVD of \( A = U \Sigma V^t \) and the desired solution is the last column of the matrix \( V \) (provided that the diagonal entries of \( \Sigma \) are arranged in descending order). Let’s prove that this solution is indeed correct. We assume that the matrix \( A \) is \( n \times m \) with \( n \geq m \). Otherwise, the solution is trivial (why?).

\[1\]These are the singular values.
G. Let $v_1, v_2, \cdots, v_m$ denote the $m$ columns of $V$. Show that any unit vector $v$ can be represented as a linear combination of $v_i$,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m,$$

with the constraint that $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_m^2 = 1$.

H. Complete the proof using the fact that $\|Av\|^2 = \alpha_1^2 \lambda_1^2 + \alpha_2^2 \lambda_2^2 + \cdots + \alpha_m^2 \lambda_m^2$, where $\lambda_i$ are the singular values. This is a simple application of the Lagrange multipliers.

**Problem 3 (Rigid Registration Problem)**

Let $p_1, \cdots, p_n$ and $q_1, \cdots, q_n$ denote two sets of $n$-points in $\mathbb{R}^3$. Suppose there exists a rigid transformation, $(R, t)$, that relates these two point sets: for $1 \leq i \leq n$,

$$q_i = R p_i + t,$$

where $R \in SO(3)$ is a rotation matrix and $t$, the translation component (vector). In most applications, the data contain noise and the equality above is rarely observed. Therefore, given two collections of points related by an unknown rigid transformation, we can estimate $(R, t)$ by solving the following least squares problem:

$$\min_{R, t} \sum_{i=1}^n |R p_i + t - q_i|^2. \quad (1)$$

A. Show that if we know the rotation matrix $R$, then the optimal translation $t$ is given by

$$t = m_q - R m_p,$$

where $m_p$ (likewise for $m_q$) is the center of the point set $p_1, \cdots, p_n$, $m_p = (p_1 + \cdots + p_n)/n$.

B. Now we proceed to solve $R$. Substituting $t = m_q - R m_p$ in Equation 1, we have

$$\min_{R} \sum_{i=1}^n |R p_i + m_q - R m_p - q_i|^2 = \sum_{i=1}^n |R (p_i - m_p) - (q_i - m_q)|^2. \quad (2)$$

This is a function of $R$ only, and if we make the substitutions $p_i = (p_i - m_p), q_i = (q_i - m_q)^3$, the error function now has the simpler form

$$\min_{R} \sum_{i=1}^n |R p_i - q_i|^2. \quad (3)$$

By expanding each summand in Equation 3, show that the sum in Equation 1 is

$$C - \alpha \sum_{i=1}^n q_i^T R p_i,$$

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$^2$This problem appeared in a qualify exam before!

$^3$This is called centering of data. We are centering the data with respect to their centers.
where $C, \alpha$ are two constants with $\alpha > 0$. (This looks complicated at first. But after a few minutes of calculation, you will be very happy and satisfied to see the result emerging.)

C. Therefore, to minimize Equation 3, we need to maximize

$$\sum_{i=1}^{n} q_i^t R p_i.$$ 

Let $P = [p_1 p_2 \cdots p_n]$ and $Q = [q_1 q_2 \cdots q_n]$, i.e, $P, Q$ are two 3-by-$n$ matrices. Show that the sum above equals to $\text{Tr}(Q^t R P)$.

Let $W = PQ^t$. We have $\text{Tr}(Q^t R P) = \text{Tr}(RPQ^t) = \text{Tr}(RW)$.

D. Finally, let $W = U^t \Sigma V$ denote the singular value decomposition of $W$. Show that $R = V^t U$ gives the desired rotation. (Recall that both $V, U$ are rotation matrices; therefore, $VV^t = V^t V = I$ (and likewise for $U$)).

**Problem 4 (Projective Geometry and Vanishing Point)**

In the previous three problems, we have used a little linear algebra and solved several important problems in vision (ellipse fitting, rigid matching, etc.). Let’s now turn to geometry.

Consider a simple perspective camera. A point $p = [x, y, z]^t \in \mathbb{R}^3$ is projected down to the image plane $(x_{im}, y_{im})$ using the formula

$$(x_{im}, y_{im}) = (\alpha \frac{x}{z}, \beta \frac{y}{z}),$$

where $\alpha, \beta > 0$ are two constants.

A. Show that for any two parallel lines $L_1, L_2 \subset \mathbb{R}^3$, the two projected lines on the image plane have an intersection point, the vanishing point. Characterize this point using the direction of the lines $L_1$ and $L_2$.

B. Prove that the vanishing points associated to three coplanar bundles of parallel lines are collinear. That is, suppose there are three collections of parallel lines on a plane in $\mathbb{R}^3$, then the three vanishing points (one vanishing point for each collection of parallel lines) are collinear.

C. Prove the orthocenter theorem: Let $T$ be the triangle on the image plane defined by the three vanishing points of three mutually orthogonal sets of parallel lines in space. The image center is the orthocenter of $T$. The orthocenter is the intersection of the three altitudes of the triangles. (This problem is rather easy. If $a, b, c$ are the three vanishing points, show that the line $aO$, where $O$ the camera center, is perpendicular to the line $bc$, etc..)