Shape from Shading

Cap5416 -- Vemuri
Perception of Shape from Shading

- Continuous image brightness variation due to shape variations is called *shading*
- Our perception of shape depends on shading
- Circular region on left is perceived as a flat disk
- Circular region on right has a varying brightness and is perceived as a sphere
Image Intensity and 3D Geometry

- Shading as a cue for shape reconstruction
- What is the relation between intensity and shape?
  - Reflectance Map
Reflectance Map

- Relates image irradiance \( I(x,y) \) to surface orientation \((p,q)\) for given source direction and surface reflectance.

- Lambertian case:
  
  \( k \) : source brightness

  \( \rho \) : surface albedo (reflectance)

  \( c \) : constant (optical system)

  Image irradiance:

  \[
  I = \frac{\rho}{\pi} k c \cos \theta_i = \frac{\rho}{\pi} k c \mathbf{n} \cdot \mathbf{s}
  \]

  Let \( \frac{\rho}{\pi} k c = 1 \) then

  \[
  I = \cos \theta_i = \mathbf{n} \cdot \mathbf{s}
  \]
Reflectance Map

- Lambertian case

\[ I = \cos \theta_i = n \cdot s = \frac{(pp_s + qq_s + 1)}{\sqrt{p^2 + q^2 + 1} \sqrt{p_s^2 + q_s^2 + 1}} = R(p, q) \]

Iso-brightness contour

Reflectance Map (Lambertian)

Cone of constant \( \theta_i \)
Reflectance Map - RECAP

- Lambertian case

Note: $R(p, q)$ is maximum when $(p, q) = (p_s, q_s)$
Shape from a Single Image?

• Given a single image of an object with known surface reflectance taken under a known light source, can we recover the shape of the object?
• Given $R(p, q)$ ($p_S, q_S$ and surface reflectance) can we determine $(p, q)$ uniquely for each image point?

NO
Solution

- Take more images
  - Photometric stereo

- Add more constraints
  - Shape-from-shading
Photometric Stereo

Lambertian case:

\[ I = \frac{\rho}{\pi} kc \cos \theta_i = \rho \mathbf{n} \cdot \mathbf{s} \quad \left( \frac{k}{\pi} = 1 \right) \]

Image irradiance:

\[ I_1 = \rho \mathbf{n} \cdot \mathbf{s}_1 \]
\[ I_2 = \rho \mathbf{n} \cdot \mathbf{s}_2 \]
\[ I_3 = \rho \mathbf{n} \cdot \mathbf{s}_3 \]

- We can write this in matrix form:

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_2
\end{bmatrix} = \rho
\begin{bmatrix}
\mathbf{s}_1^T \\
\mathbf{s}_2^T \\
\mathbf{s}_3^T
\end{bmatrix}
\mathbf{n}
\]
Solution

- Take more images
  - Photometric stereo

- Add more constraints
  - Shape-from-shading (this class)
Human Perception

• Our brain often perceives shape from shading.

• Mostly, it makes many assumptions to do so.

• For example:

  Light is coming from above (sun).

  Biased by occluding contours.

by V. Ramachandran
See Ramachandran’s work on Shape from Shading by Humans

Problem

$(p,q)$ can be infinite when $\theta = 90^\circ$

Redefine reflectance map as $R(f, g)$

\[
f = \frac{2p}{1 + \sqrt{1 + p^2 + q^2}} \quad \text{and} \quad g = \frac{2q}{1 + \sqrt{1 + p^2 + q^2}}
\]
Occluding Boundaries

\[ \mathbf{n} \perp \mathbf{e}, \quad \mathbf{n} \perp \mathbf{v} \quad \therefore \quad \mathbf{n} = \mathbf{e} \times \mathbf{v} \quad \mathbf{e} \text{ and } \mathbf{v} \text{ are known} \]

The \( \mathbf{n} \) values on the occluding boundary can be used as the boundary condition for shape-from-shading.
Image Irradiance Constraint

• Image irradiance should match the reflectance map

Minimize

\[ e_i = \int \int_{\text{image}} (I(x, y) - R(f, g))^2 \, dx \, dy \]

(minimize errors in image irradiance in the image)
Smoothness Constraint

- Used to constrain shape-from-shading
- Relates orientations \((f,g)\) of neighboring surface points

Minimize

\[
e_s = \iint_{\text{image}} \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) \, dx \, dy
\]

\((f, g)\): surface orientation under stereographic projection

\[
f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad g_x = \frac{\partial g}{\partial x}, \quad g_y = \frac{\partial g}{\partial y}
\]

(penalize rapid changes in surface orientation \(f\) and \(g\) over the image)
Shape-from-Shading

- Find surface orientations \((f, g)\) at all image points that minimize

\[
e = e_s + \lambda e_i
\]

Minimize

\[
e = \iint_{\text{image}} \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) + \lambda (I(x, y) - R(f, g))^2 \, dx \, dy
\]
Numerical Shape-from-Shading

- **Smoothness error** at image point \((i,j)\)

\[
s_{i,j} = \frac{1}{4} \left( (f_{i+1,j} - f_{i,j})^2 + (f_{i,j+1} - f_{i,j})^2 + (g_{i+1,j} - g_{i,j})^2 + (g_{i,j+1} - g_{i,j})^2 \right)
\]

Of course you can consider more neighbors (smoother results)

- **Image irradiance error** at image point \((i,j)\)

\[
r_{i,j} = \left( I_{i,j} - R(f_{i,j}, g_{i,j}) \right)^2
\]

Find \(\{f_{i,j}\}\) and \(\{g_{i,j}\}\) that minimize

\[
e = \sum_i \sum_j \left( s_{i,j} + \lambda r_{i,j} \right)
\]

(Ikeuchi & Horn 89)
Find \( \{f_{i,j}\} \) and \( \{g_{i,j}\} \) that minimize \( e = \sum_{i} \sum_{j} (s_{i,j} + \lambda r_{i,j}) \)

If \( f_{k,l} \) and \( g_{k,l} \) minimize \( e \), then \( \frac{\partial e}{\partial f_{k,l}} = 0, \frac{\partial e}{\partial g_{k,l}} = 0 \)

\[
\frac{\partial e}{\partial f_{k,l}} = 2(f_{k,l} - \bar{f}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial f} \right|_{f_{k,l}} = 0
\]

\[
\frac{\partial e}{\partial g_{k,l}} = 2(g_{k,l} - \bar{g}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial g} \right|_{g_{k,l}} = 0
\]

where \( \bar{f}_{k,l} \) and \( \bar{g}_{k,l} \) are 4-neighbors average around image point \( (k,l) \)

\[
\bar{f}_{k,l} = \frac{1}{8} (f_{i+1,j} + f_{i,j+1} + f_{i-1,j} + f_{i,j-1})
\]

\[
\bar{g}_{k,l} = \frac{1}{8} (g_{i+1,j} + g_{i,j+1} + g_{i-1,j} + g_{i,j-1})
\] (Ikeuchi & Horn 89)
Numerical Shape-from-Shading

\[ \frac{de}{df_{k,l}} = 2(f_{k,l} - \bar{f}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \frac{\partial R}{\partial f}_{f_{k,l}} = 0 \]

\[ \frac{de}{dg_{k,l}} = 2(g_{k,l} - \bar{g}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \frac{\partial R}{\partial g}_{g_{k,l}} = 0 \]

**Update rule**

\[ f_{k,l}^{n+1} = \bar{f}_{k,l} + \lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \frac{\partial R}{\partial f}_{f_{k,l}} \]

\[ g_{k,l}^{n+1} = \bar{g}_{k,l} + \lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \frac{\partial R}{\partial g}_{g_{k,l}} \]

- Use known \((f, g)\) values on the occluding boundary to constrain the solution (boundary conditions)
- Compare \((f_{k,l}^{n+1}, g_{k,l}^{n+1})\) with \((f_{k,l}^{n}, g_{k,l}^{n})\) for convergence test
- As the solution converges, increase \(\lambda\) to remove the smoothness constraint

(Ikeuchi & Horn 89)
Calculus of Variations

Minimize

\[ e = \iint_{\text{image}} F(f, g, f_x, f_y, g_x, g_y) \, dx \, dy \]

\[ F = \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) + \lambda \left( I(x, y) - R(f, g) \right)^2 \]

Euler equations for \( F \)

\[ F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0, \quad F_g - \frac{\partial}{\partial x} F_{g_x} - \frac{\partial}{\partial y} F_{g_y} = 0 \]

(Read Horn A.6)

Euler equations for shape-from-shading

\[ \nabla^2 f = -\lambda \left( I(x, y) - R(f, g) \right) \frac{\partial R}{\partial f}, \quad \nabla^2 g = -\lambda \left( I(x, y) - R(f, g) \right) \frac{\partial R}{\partial g} \]

Solve this coupled pair of second-order partial differential equations with the occluding boundary conditions!
Results

by Ikeuchi and Horn
Results

Scanning Electron Microscope image (inverse intensity)

by Ikeuchi and Horn