Projective Geometry

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Projective Geometry

1. Solving a Homogeneous Linear System of Equations

2. A Hierarchy of Transformations
Homogeneous Linear Systems

1. Let $A \in \mathbb{R}^{(m \times n)}$ and $x \in \mathbb{R}^n$, $\text{Rank}(A) = (n - 1)$, find the non-trivial solution for $Ax = 0$. Note, the trivial solution is $x = 0$.

2. A solution unique up to a scale factor is easily found by using SVD decomposition.

3. **Theorem**: This solution is proportional to the eigen-vector corresponding to the only zero eigen value of $A^tA$ (all other eigenvalues are strictly positive, why?).
Look for a unit-norm solution in the least-squares sense, since norm of the solution otherwise can be arbitrary.

Therefore, minimize \( \| Ax \|^2 = (Ax)^t Ax = x^t A^t Ax \), subject to \( x^t x = 1 \). Using Lagrange multiplier \( \lambda \), this is equivalent to minimizing,

\[
\mathcal{L}(x) = x^t A^t Ax - \lambda(x^t x - 1) \tag{1}
\]

Taking the gradient of the above eqn. and equating to zero, we get \( A^t Ax - \lambda x = 0 \).

\[ \Rightarrow \lambda \text{ is the eigen value of } A^t A \text{ and } x = e_{\lambda} \text{ the corresponding eigen vector.} \]
Proof (Contd.)

- Replacing $\mathbf{x}$ with $\mathbf{e}_\lambda$ and $A^t A \mathbf{e}_\lambda$ with $\lambda \mathbf{e}_\lambda$ in 1, we get
- $\mathcal{L}(\mathbf{e}_\lambda) = \lambda \Rightarrow$ minimum is reached at $\lambda = 0$, the least eigenvalue of $A^t A$.
- Thus the solution is the column of $V$ in the SVD of $A$ that corresponds to the only singular value of $A$. Why columns of $V$? This completes the proof.
Let $A \in \mathbb{R}^{m \times n}$, $U \in \mathbb{R}^{m \times m}$ of Orthogonal matrices, 
$\Sigma \in \mathbb{R}^{m,n}$ be diagonal containing the singular values, and 
$V \in \mathbb{R}^{n \times n}$ of Orthogonal matrices, then, $A = U\Sigma V^t$, with 
$\sigma_1 \geq \sigma_2 \cdots \geq 0$. $U$ and $V$ have unit norm columns.

**Proposition:** Columns of $U$ corresponding to the nonzero singular values span the range of $A$, columns of $V$ corresponding to the zero singular values span the null space of $A$. 
Class-1: Isometries

- Transforms that preserve distances in the plane $\mathbb{R}^2$. Example: Rotation, rigid (rotation+translation) transformation.

\[
\begin{pmatrix}
x' \\
y' \\
1
\end{pmatrix} =
\begin{pmatrix}
\varepsilon \cos \theta & -\sin \theta & t_x \\
\varepsilon \sin \theta & \cos \theta & t_y \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
\] (2)

- $\varepsilon = 1 \Rightarrow$ orientation preserving, $\varepsilon = -1 \Rightarrow$ orientation reversing (reflection).
Rigid Motion

- Can write 2 as

\[ x' = H_e x = \begin{pmatrix} R & t \\ 0_t & 1 \end{pmatrix} x \]

- \( R : (2, 2) \) matrix; \( R^t R = I = R R^t \) \( t = (2, 1) \)-vector; \( \mathbf{0} : (2, 1) \) null-vector.
- \( \text{DOF: 3; can compute } H_e \text{ from 2-point correspondences.} \)
- \textbf{Invariants:} lengths, angle and area.
Class-II: Planar Similarity

- **Isotropic scaling:**

\[
\begin{pmatrix}
  x' \\
  y' \\
  1
\end{pmatrix} =
\begin{pmatrix}
  s \cos \theta & -s \sin \theta & t_x \\
  s \sin \theta & s \cos \theta & t_y \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  1
\end{pmatrix}
\]  
\hspace{1cm} (3)

- **Consisely,**

\[
x' = H_s x = \begin{pmatrix} sR & t \\ 0 & 1 \end{pmatrix} x
\]  
\hspace{1cm} (4)

Where, \( s \) is an isotropic scaling factor.

- **DOF:** 4, can be computed from 2 point correspondences.

- **Invariants:** (i) Angles between lines, (ii) ratio of two lengths (iii) ratio of areas.
Class-III: Affine Transforms

- These are non-singular linear transforms followed by translation.

\[
\begin{pmatrix}
x' \\
y' \\
1
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} & t_x \\
a_{21} & a_{22} & t_y \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
\]

(5)

- Consisely,

\[
x' = H_a x = \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} x
\]

(6)

A is a non-singular linear transformation.

- Can represent \( A = R(\theta)R(-\phi)DR(\phi) \), why? Hint:......
Affine Transform (Contd.)

- **Invariants:** (i) Parallel lines map to parallel lines.
  - Two parallel lines intersect at an ideal point \((x_1, x_2, 0)^t\).
  - Under an Affine transform, it is mapped to another ideal point \(\Rightarrow\) parallel lines remain parallel.
- Ratio of lengths of parallel line segments is unchanged (prove it).
- Ratio of areas is unchanged (prove it).
Projective Transformations

- It is a general non-singular linear transform of homogeneous coordinates. $x' = H_px$.

\[
x' = \begin{pmatrix} A & t \\ v^t & v \end{pmatrix} x
\]  

(7)

- Where, $v = (v_1, v_2)^t$. It is not in general possible to scale the matrix to make $v = 1$ as $v$ could be 0.

- Note that Affine transforms map ideal points to ideal points but projective transforms DON'T.
Deomposition of Projective Transforms

- Projective transforms can be decomposed as:

\[ H = H_s H_a H_p = \begin{pmatrix} sR & t \\ 0^t & 1 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0^t & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ v^t & v \end{pmatrix} = \begin{pmatrix} A & t \\ v^t & v \end{pmatrix} \] (8)

Where, \( A = sRK + tv^t \), \( K \) is an upper triangular matrix with \( \text{det}(K) = 1 \). Decomposition is valid if \( v \neq 0 \) and unique if \( s > 0 \).

- DOF = 8.
Invariants of Projective Transform

Cross Ratio is the ratio that is preserved between two sets of points that differ by a projectivity (projective transform).

\[ \text{Cross}(x_1', x_2', x_3', x_4') = \text{Cross}(x_1, x_2, x_3, x_4) \] (9)

\[ \text{Cross}(x_1, x_2, x_3, x_4) = \frac{|x_1 x_2||x_3 x_4|}{|x_1 x_3||x_2 x_4|} \] (10)

\[ |x_i x_j| = \det \begin{pmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{pmatrix} \] (11)

Figure: Cross Ratio as an Invariant
Solving a Homogeneous Linear System of Equations

A Hierarchy of Transformations

Back to DLT and Variants

- In DLT, we minimize $||Ah||^2$ and SVD to solve the ensuing homogenous linear system. This is the “Algebraic distance” as a cost function,
  
  $$d_{alg}(x'_i, Hx_i) = ||\epsilon_i||^2 = ||A_i h||^2.$$

- Geometric distance: also called transfer error (we fix one of the images as a calibration pattern where measurements are highly accurate) –
  
  $$\arg\min_H \sum_i (d(x_i, H^{-1}x'_i))^2$$

- Symmetric transfer error:
  
  $$\arg\min_H \sum_i (d(x_i, H^{-1}x'_i))^2 + \sum_i (d(x'_i, Hx_i))^2.$$
Reprojection Error

- Points $\mathbf{x}$ and $\mathbf{x}'$ are measured noisy points.
- Under estimated homography, $\mathbf{x}'$ and $H\mathbf{x}$ do not correspond perfectly and neither do $\mathbf{x}$ and $H^{-1}\mathbf{x}'$.
- However, $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ match perfectly via $\hat{\mathbf{x}}' = \hat{H}\hat{\mathbf{x}}$.
- Hence we want to minimize:

$$\arg\min_{H, \hat{\mathbf{x}}', \hat{\mathbf{x}}_i} \sum_i (d(\mathbf{x}_i, \hat{\mathbf{x}}_i))^2 + (d(\mathbf{x}'_i, \hat{\mathbf{x}}'_i))^2$$

Such that, $\hat{\mathbf{x}}'_i = \hat{H}\hat{\mathbf{x}}_i \cdot \cdot \cdot \forall i$. 