A parameterized surface $X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a differentiable map \(^1\) $X$ from an open set $U \in \mathbb{R}^2$ to $\mathbb{R}^3$.

Explicitly, $X(U)$ is written as

$$X(u, v) = (x(u, v), y(u, v), z(u, v)).$$

$X(U) \in \mathbb{R}^3$ is called the trace of $X$.

\(^1\)X(u, v) is called differentiable if $x(u, v)$, $y(u, v)$ and $z(u, v)$ are differentiable functions w.r.t. the parameters $u$, $v$. 
Parameterizations of a sphere - examples

Figure: (a) Sphere in $\mathbb{R}^3$. (b) Six overlapping parameterization of a kind $X(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$. (c) $X(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.

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$^2$Differential geometry of curves and surfaces, by Manfredo P. do Carmo
Differential of $X$

Let $X : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differential map. For each $p \in U$ we will define the differential of $X$ at $p$ by

$$dX_p : U \subset \mathbb{R}^n \to \mathbb{R}^m$$

Let $w \in \mathbb{R}^n$ and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differential curve such that $\alpha(0) = p$, $\alpha'(0) = w$.

An image of $\alpha$ in $\mathbb{R}^m$ is given by

$$\beta = X \circ \alpha(-\epsilon, \epsilon) \to \mathbb{R}^m$$

Then:

$$dX_p(w) = \beta'(0)$$
Differential of $X$: illustration
Calculation of the differential in $\mathbb{R}^3$

Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a differential map.

$$\alpha(t) = (u(t), v(t))$$

$$\beta(t) = X \circ \alpha(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$$

$$\beta'(0) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = dX_p(w)$$

Note: $dX_p(w)$ depends only on $w$, and not on the choice of $\alpha(t)$. 
Exercise: $dF_p$ calculation

Calculate $dF_p$ for $F(u, v) = (u^2 - v^2, 2uv)$, $(u, v) \in \mathbb{R}^2$.

Solution:

\[ x = u^2 - v^2, \quad y = 2uv \]

\[
dF_p = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix} = \begin{pmatrix}
2x & -2y \\
2y & 2x
\end{pmatrix}
\]

\[
dF_{(1,1)}(2, 3) = dF_p \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (-2, 10)
\]
We will say that a differential parameterization $X(U)$ is *regular* if $X_u(u, v)$ and $X_v(u, v)$ are linearly independent for all $(u, v) \in U$.

*Tangent plane (space) at $p$: a plane $T_p$ containing all the tangents to curves passing through the point $p \in S$. The tangent plane is spanned by the vectors $X_u(u, v), X_v(u, v)$.*

*Surface normal: a unit vector orthogonal to the tangent plane at the point $p$, defined by*

$$N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p)$$
Surfaces in $\mathbb{R}^3$ - examples
Exercise: tangent space, normal

Calculate the equation of the tangent plane of a surface which is a graph of a differentiable function \( z = f(x, y) \), at the point \((x_0, y_0)\).

Solution:

\[
X(x, y) = (x, y, f(x, y))
\]

\[
X_x(x, y) = (1, 0, f_x(x, y))
\]

\[
X_y(x, y) = (0, 1, f_y(x, y))
\]

\[
N = \frac{X_u \times X_v}{\| X_u \times X_v \|} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}
\]

Equation of a plane passing through \((x_0, y_0)\) and perpendicular to \(N(x_0, y_0)\) is:

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0
\]
The first fundamental form

In order to measure local distances on the regular surface $S$, we assign an inner product on the tangent plane $T_p$ of $S$ at every point $p$.

The first fundamental form: a quadratic form $I_p : T_p(S) \times T_p(S) \to \mathbb{R}$, given by

$$I_p(w_1, w_2) = \langle w_1, w_2 \rangle_p.$$ 

$\langle \cdot, \cdot \rangle_p$ is a standard inner product of $\mathbb{R}^3$; $p$ stands for the tangent plane $T_p$ of the surface $S$ at the point $p$.

The form $I_p(w_1, w_2)$ is bilinear, symmetric and positive-definite.
The first fundamental form in terms of $X_u, X_v$

Let $\alpha_1(t) = X(u_1(t), v_1(t))$ and $\alpha_2(t) = X(u_2(t), v_2(t))$ be two differential curves such that

$$\alpha_1(0) = \alpha_2(0) = p, \quad \alpha'_1(0) = w_1, \quad \alpha'_2(0) = w_2.$$ 

The first fundamental form is given by:

$$l_p(w_1, w_2) = \langle \alpha'_1(0), \alpha'_2(0) \rangle = \langle X_u u'_1 + X_v v'_1, X_u u'_2 + X_v v'_2 \rangle$$

$$= \begin{pmatrix} u'_1 \\ v'_1 \end{pmatrix}^T \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix} \begin{pmatrix} u'_2 \\ v'_2 \end{pmatrix}$$

$$= \begin{pmatrix} u'_1 \\ v'_1 \end{pmatrix}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u'_2 \\ v'_2 \end{pmatrix}$$
The first fundamental form - alternative notation

\[
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}
= \begin{pmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{pmatrix}
= \begin{pmatrix}
\langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\
\langle X_v, X_u \rangle & \langle X_v, X_v \rangle
\end{pmatrix}
\]

In coordinate notation:

\[
l_p(w_1, w_2) = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij} \tilde{w}_1^i \tilde{w}_2^j = g_{ij} \tilde{w}_1^i \tilde{w}_2^j
\]

where we use the *Einstein summation convention*, according to which identical sub- and super-scripts are summed up; \( \tilde{w}_k = (u'_k, v'_k), k = 1, 2 \).
Exercise: calculation of $I_p$

Calculate the first fundamental form of a unit sphere parameterized by

$$X(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

Solution:

$$X_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$$

$$X_\phi = (\sin \theta - \sin \phi, \sin \theta \cos \phi, 0)$$

$$E = \langle X_\theta, X_\theta \rangle = \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta = 1$$

$$F = \langle X_\theta, X_\phi \rangle = -\cos \theta \cos \phi \sin \theta \sin \phi + \cos \theta \sin \phi \sin \theta \cos \phi = 0$$

$$G = \langle X_\phi, X_\phi \rangle = \sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi = \sin^2 \theta$$
Length calculation

The length of a parameterized curve $\beta : [0, T] \rightarrow S$ is given by

$$s(t) = \int_0^t \|\beta'(\tilde{t})\| d\tilde{t} = \int_0^t \sqrt{I_p(\beta', \beta')} d\tilde{t}$$

Given that $\alpha(t) = (u(t), v(t))$, $\beta(t) = X \circ \alpha(t)$,

$$s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} d\tilde{t}$$

Also, local element of the arclength is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$
Area calculation

Area of a bounded region $R$ of a regular surface $S = X(U)$ is given by

$$A(R) = \int \int_Q \|X_u \times X_v\| \ dudv, \quad Q = X^{-1}(R)$$

From the equality

$$\|X_u \times X_v\|^2 + \langle X_u, X_v \rangle^2 = \|X_u\|^2 \|X_v\|^2$$

it follows that

$$\|X_u \times X_v\| = \sqrt{EG - F^2}$$

Note that $EG - F^2 > 0$, by definition (prove).
Exercise: arclength and area measured on a sphere

Calculate $ds^2$ and $dA = \|X_u \times X_v\| \, d\theta d\phi$ of a unit sphere parameterized by

$$X(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

Solution: the first fundamental form of a sphere is given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$

Therefore:

$$ds^2 = Ed\theta^2 + 2Fd\theta d\phi + Gd\phi^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

$$dA = \sqrt{EG - F^2} \, d\theta d\phi = \sin^2 \theta \, d\theta d\phi$$
We saw that a surface normal at point \( p \in S \) is defined by:

\[
N(p) = \frac{X_u \times X_v}{\|X_u \times X_v\|}(p)
\]

We will say that a regular surface \( S \) is *orientable* if and only if there exist a differentiable field of unit normal vectors \( N : S \rightarrow \mathbb{R}^3 \) on \( S \).

Orientable surfaces: sphere, paper roll.
Non-orientable surface: *Möbius* strip.