1. Find the curve $f(x)$ that minimizes the integral
\[
\int_1^2 \frac{\sqrt{1+f''^2}}{x} \, dx, \quad \text{with} \quad f(1) = 0, \quad \text{and} \quad f(2) = 1.
\]

2. Given a curve $y = f(x)$,
\[
\text{Minimize} \int_{-a}^a f(x) \, dx \quad \text{subject to} \quad f(-a) = f(a) = 0 \quad \text{and} \quad \int_{-a}^a \sqrt{1+(f'(x))^2} \, dx = l.
\]

3. Given $n$ data points $\{x_i, y_i\}_{i=1,...,n}$ such that $f(x_i) = y_i$, how many unknowns are there if a 4th order (quartic) spline is to be fitted to this data? Deduce the conditions required to solve the unknowns. Do you have enough conditions? If not, what conditions can you impose?

4. Given a set of sample measurements of a one dimensional curve in the image plane, $f(x)$, what is the purpose of minimizing the following functional:
\[
E(S) = \int \left\{ (S''(x))^2 + \lambda (f(x) - S(x))^2 \sum_k \delta(x - x_k) \right\} \, dx
\]
Describe the significance of each term on the right hand side of the above equation. Assume that $\lambda$ is a regularization parameter proportional to the known noise variance in the data. Then, write down the Euler-Lagrange equation for this minimization problem.

5. We know that the length element of a curve $y = f(x)$ is given by $ds = \sqrt{1+(f'(x))^2} \, dx$. Show that the area element of the image surface $I(x, y)$ is given by $dA = \sqrt{1+I_x^2 + I_y^2} \, dxdy$, where $I_x$ and $I_y$ are the partial derivatives of $I(x, y)$ with respect to $x$ and $y$ respectively. Next, find the Euler-Lagrange equations for the functional $\int dA$. 

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