PageRank as a Function of the Damping Factor

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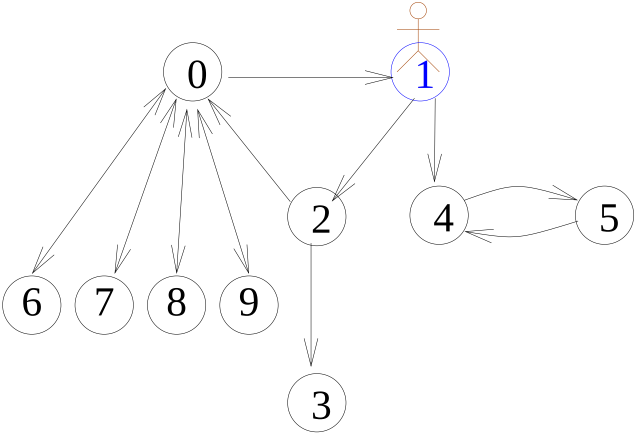
PageRank [Page et al., '98]: the Web-Surfer Metaphor



A surfer is wandering about the web. . .

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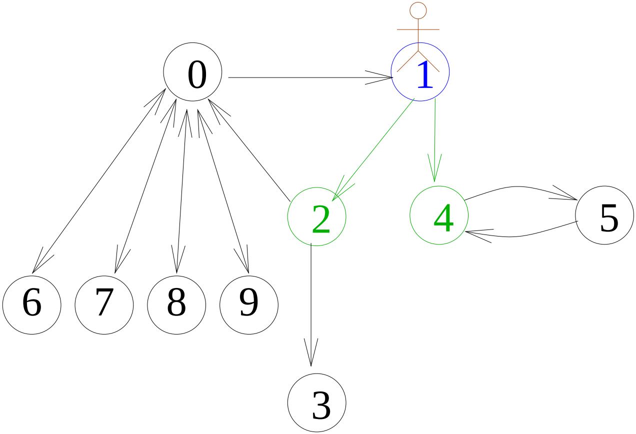

At each step, with probability α (s)he chooses the next page by clicking on a random link. . .

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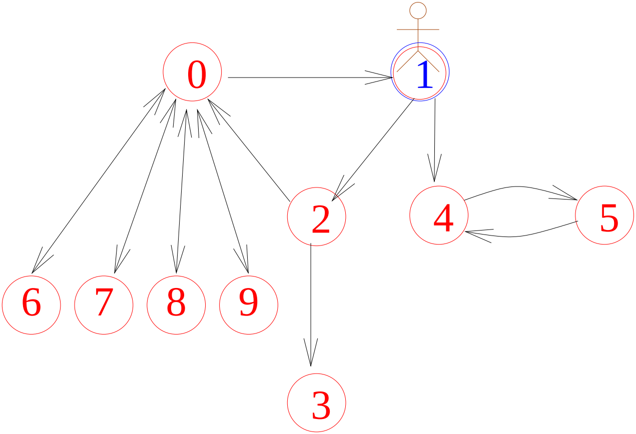

...with probability $1 - \alpha$, (s)he jumps to a random node (chosen uniformly or according to a fixed distribution, the *preference vector*)

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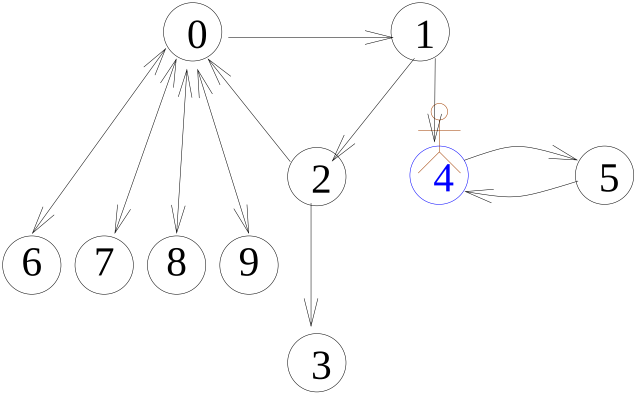

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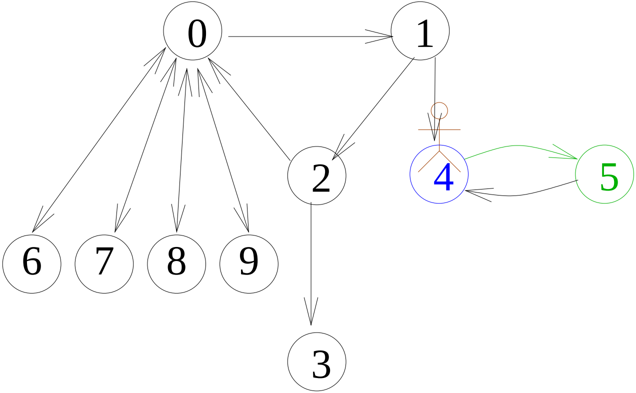

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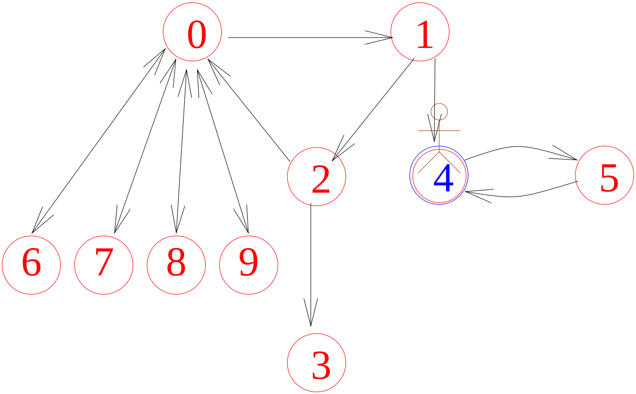

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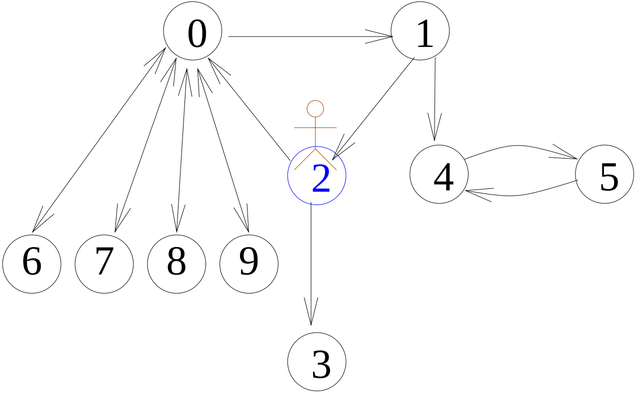

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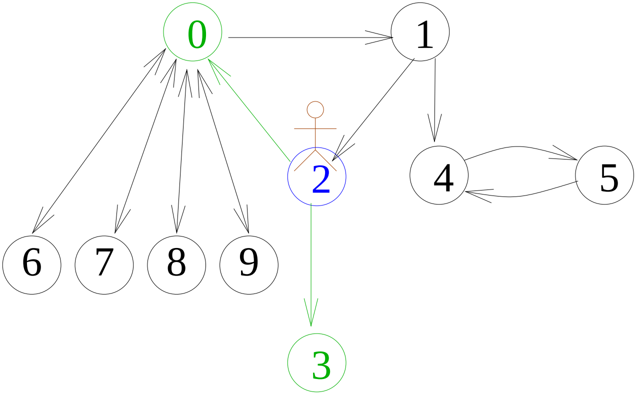

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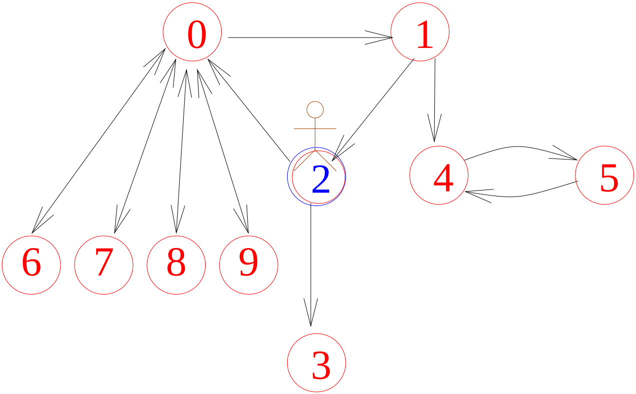

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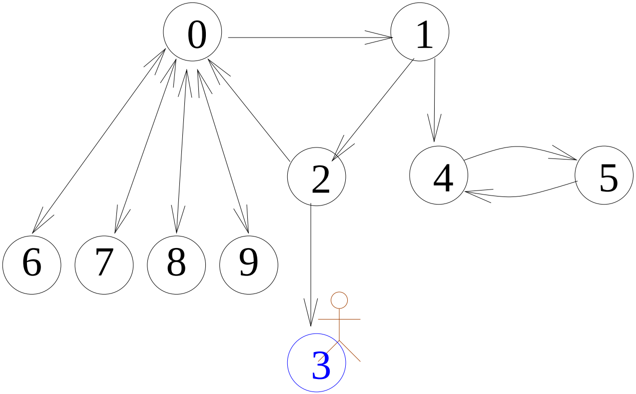

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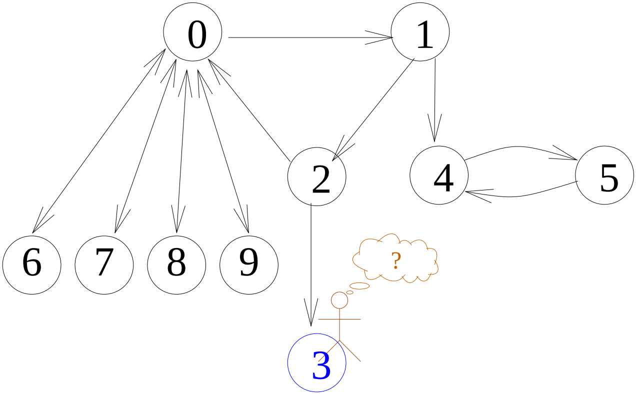

What if (s)he reaches a node with no outlinks (a *dangling node*)?

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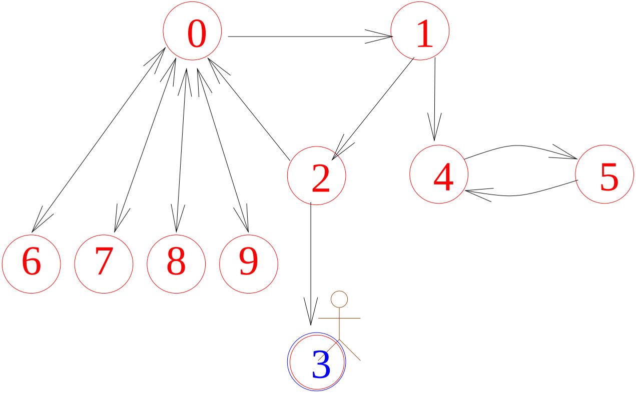

In that case, (s)he jumps to a random node *with probability 1*.

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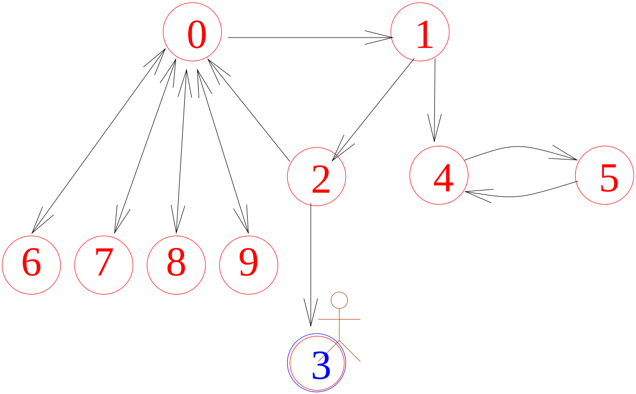

The PageRank of a page is the average fraction of time spent by the surfer on that page.

What does PageRank depends on?

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- the *preference vector* $\mathbf{v}$
- the *damping factor* α.

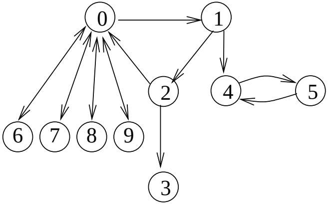
We will focus on the

damping factor

Notation

G = the (adjacency matrix of the) graph

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Notation

$\bar{G}$ = the modified graph to eliminate dangling nodes

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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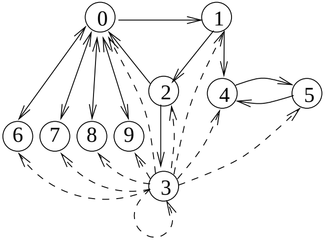
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$



Notation

P = the row-normalized version of $\bar{G}$

$$
\begin{pmatrix}
0 & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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$$

Notation

$$A(\alpha) = \alpha P + (1-\alpha)\mathbf{1}^T\mathbf{v}$$

$$\begin{pmatrix}
\frac{1-\alpha}{10} & \frac{\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{\alpha+1}{10} & \frac{\alpha+1}{10} & \frac{\alpha+1}{10} & \frac{\alpha+1}{10} \\
\frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{4\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{4\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} \\
\frac{4\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{4\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} \\
\frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\
\frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{9\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} \\
\frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{9\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} \\
\frac{9\alpha+1}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} & \frac{1-\alpha}{10} \\
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\end{pmatrix}$$

(Here, we assumed $\mathbf{v} = (1/N)\mathbf{1}$)

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$\mathbf{r}(\alpha)$ is the PageRank vector, as a (vector) function of α:

$$\mathbf{r} : [0, 1) \rightarrow [0, 1]^n.$$

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- ...numeric instability arises when α is too close to 1...

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- Basic knowledge about PageRank nature.
- Variations of PageRank to obtain better/alternative ranking techniques (such as TotalRank [Boldi, '05]).
- Tool for link-spam detection [Zhang et al., '04].

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- The condition number of the PageRank problem is $(1+\alpha)/(1-\alpha)$ [Haveliwala & Kamvar, '03]

Explicit formula for PageRank

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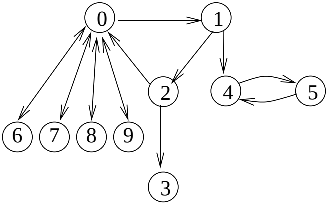
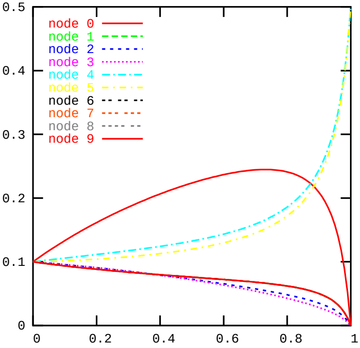
...it can be extended by continuity to $[0, 1]$.

Our example





$$r_0(\alpha) = -5\,\frac{(-1+\alpha)\left(\alpha^2+18\,\alpha+4\right)}{8\,\alpha^4+\alpha^3-170\,\alpha^2-20\,\alpha+200}$$

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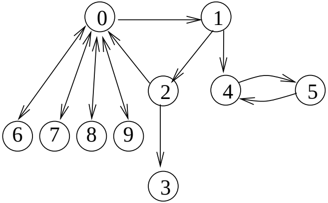

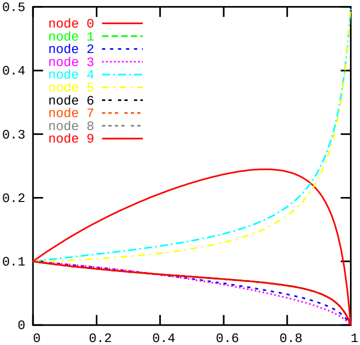

$$r_1(\alpha) = -2\,\frac{(-1+\alpha)\left(\alpha^2+2\,\alpha+10\right)}{8\,\alpha^4+\alpha^3-170\,\alpha^2-20\,\alpha+200}$$

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It is not *unique*, though, unless P itself is aperiodic and irreducible.

From the point of view of the Markov chain

We would like to compute

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- on the other hand, $P = \lim_{\alpha \to 1^-} A(\alpha)$ can be reducible, so $\lim_{n\to\infty} P^n$ needs not to be stable.

Some technicalities...

For a generic Markov chain P, the limit behaviour of P^n is

$$\lim_{n\to\infty} (P^{(nd_j+r)})_{i,j} = \pi^{(r)}_{i,j}$$

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At least componentwise, these are converging subsequences, hence it is possible to define the *Cesaro* limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{0<t\le n} P^t = \Pi, \qquad (\Pi)_{i,j} = \frac{1}{d_j} \sum_{1\le r\le d_j} \pi^{(r)}_{i,j}.$$

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By the limiting behaviour of subsequences and the Cesaro limit

$$\lim_{\alpha\to 1^-}\lim_{n\to\infty} A(\alpha)^n = \mathbf{1}^T\mathbf{v}\Pi$$

Back to the limit

Two easy Lemmas:

- $\lim_{n\to\infty} A(\alpha)^n = \mathbf{1}^T \mathbf{v}(1-\alpha)\sum_{n\geq 0}(\alpha P)^n$
- $(1-\alpha)\sum_{n\geq 0}\alpha^n a_n \sim a$ for every $a_n \to a$ (as $\alpha \to 1^-$)

By the limiting behaviour of subsequences and the Cesaro limit

$$\lim_{\alpha\to 1^-}\lim_{n\to\infty} A(\alpha)^n = \mathbf{1}^T \mathbf{v}\Pi$$

or, from the point of view of the limit distribution,

$$\mathbf{r}^* = \lim_{\alpha\to 1^-}\lim_{n\to\infty} \mathbf{p}_0 A(\alpha)^n = \mathbf{v}\Pi.$$

(whatever the initial $\mathbf{p}_0$).

So what are the consequences on the limit probability?

To understand what happens when $\alpha \to 1$, we can consider an (arbitrary) stationary distribution $\mathbf{t}$ of P.

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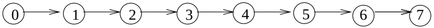
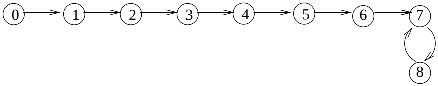
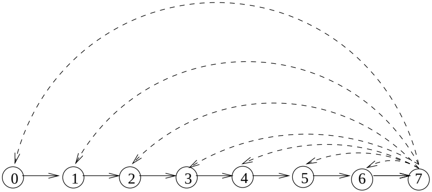
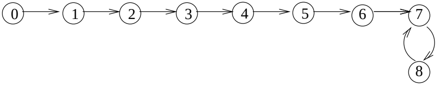
So what are the components of $\bar{G}$ with no outgoing arcs?

Components of $\bar{G}$

G_1	
G_2	

Two graphs: they look quite similar, but. . .

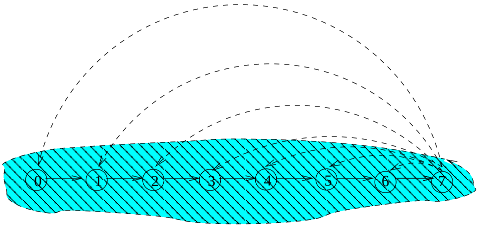
Components of $\bar{G}$

$\bar{G}_1$	
$\bar{G}_2$	

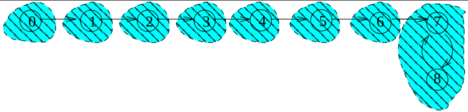
... G_1 has a dangling node, whereas G_2 has none

Components of $\bar{G}$

components of $\bar{G}_1$



components of $\bar{G}_2$



All nodes are recurrent in G_1, whereas all nodes except for 7 and 8 are transient in G_2

A general statement

Theorem

Except for degenerate cases a node is recurrent iff it belongs to a component of G that is nontrivial and has no arcs going out of it.

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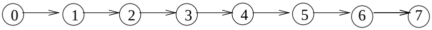
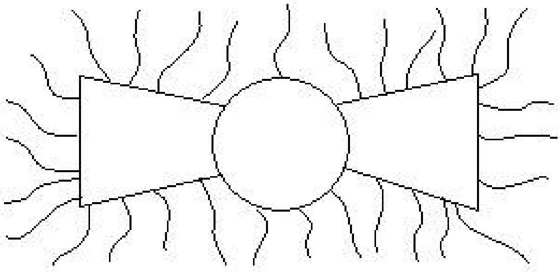
The degenerate cases happen when there are no components satisfying the above conditions, in which case *every node is recurrent*:



Bowtie

As a consequence, when $\alpha \to 1$, all PageRank concentrates in a bunch of pages that live in the rightmost part of the bowtie [Kumar et al., '00]:



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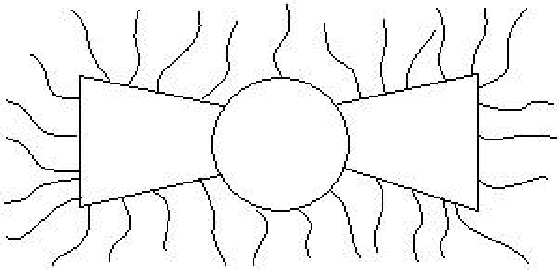

$\mathbf{r}(\alpha)$ becomes meaningless as $\alpha \to 1$!

General behaviour

What about the general behaviour of $\mathbf{r}(\alpha)$?

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The following theorem gives an explicit formula for derivatives of PageRank:

Theorem

For every $k > 0$

$$\mathbf{r}^{(k)}(\alpha) = k!\,\mathbf{r}(\alpha)\left(Q(\alpha) - \frac{1}{1-\alpha}I\right)Q(\alpha)^{k-1}.$$

where

$$Q(\alpha) = P(I - \alpha P)^{-1}.$$

Computing the derivatives. . .

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There exists an algorithm that simultaneously approximates $\mathbf{r}(\alpha)$, $\mathbf{r}'(\alpha)$, $\mathbf{r}''(\alpha)$, $\ldots$, $\mathbf{r}^{(k)}(\alpha)$ *(for a fixed value of* $\alpha \in [0, 1)$*).*

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This algorithm provides *pointwise approximation for arbitary derivatives* with precision guarantee.

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or, equivalently,

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Maclaurin expansion

Hence, the Maclaurin expansion of $\mathbf{r}(\alpha)$, that is $\mathbf{r}(\alpha) = \sum_{k=0}^{\infty} \frac{\mathbf{r}^{(k)}(0)}{k!}\alpha^k$ can be written as

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Then, for all k,

$$\mathbf{v}\left(P^k - P^{k-1}\right) = \frac{R_k - R_{k-1}}{\alpha_0^k}.$$

(by convention, $R_{-1} = \mathbf{0}$)

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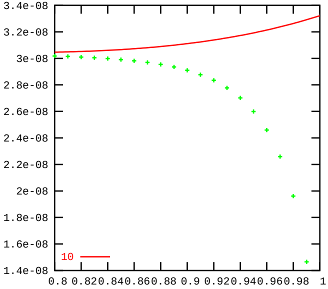

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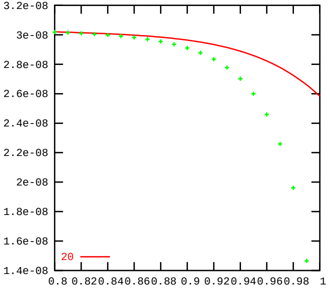

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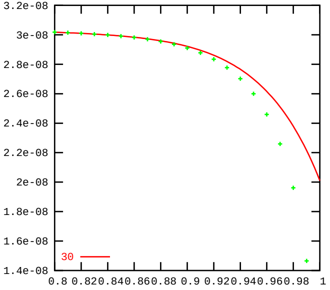

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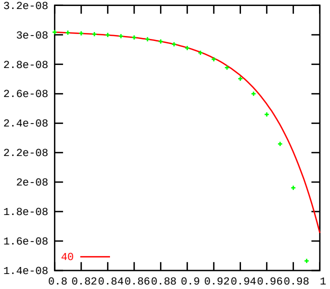

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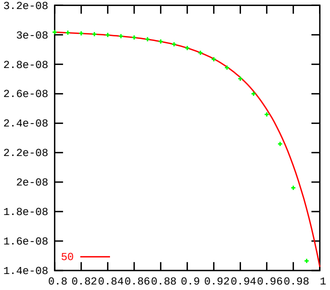

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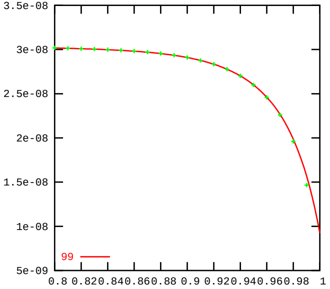
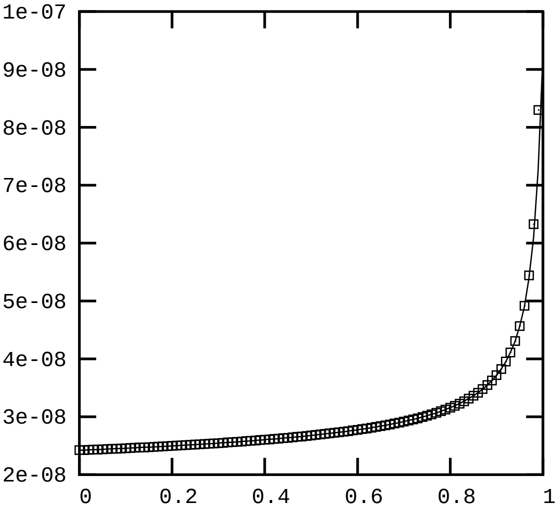


degree 99

Some typical behaviours



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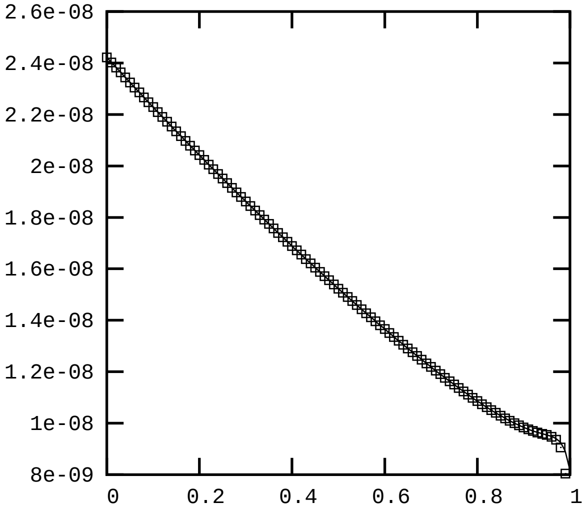

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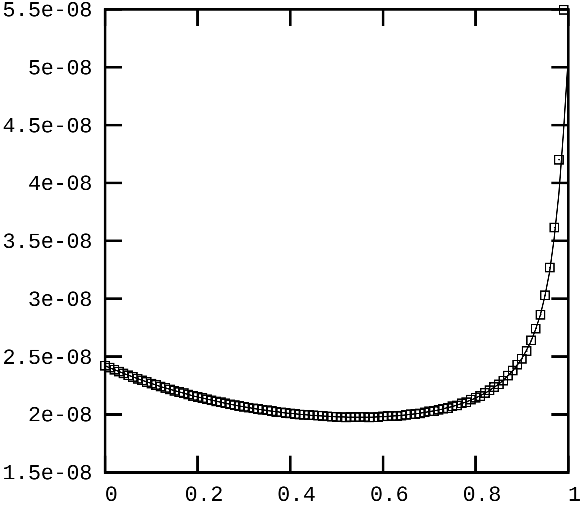

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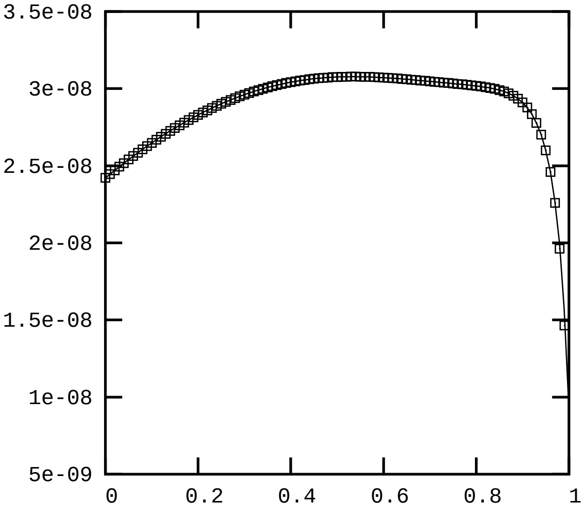

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The "Laboratory for Web Algorithmics"

`http://law.dsi.unimi.it/`

provides (free) datasets and (GNU GPL) code for dealing with large web graphs and compute PageRank and friends. . .