

# COT5520: COMPUTATIONAL GEOMETRY

## Homework # 3

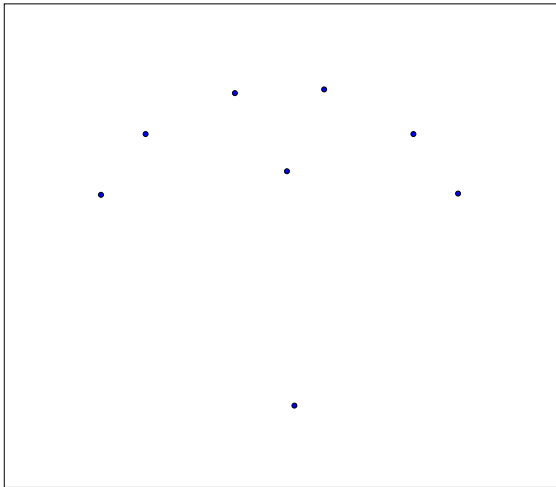
**Due date:** Nov 11, 2008, Tuesday (beginning of the class)

Your solutions should be concise, but complete, and typed (or handwritten clearly). Feel free to consult textbooks, journal and conference papers and also each other, but write the solutions yourself and cite your sources. Answer **only five** of the following six questions. Each problem is worth 20 pts.

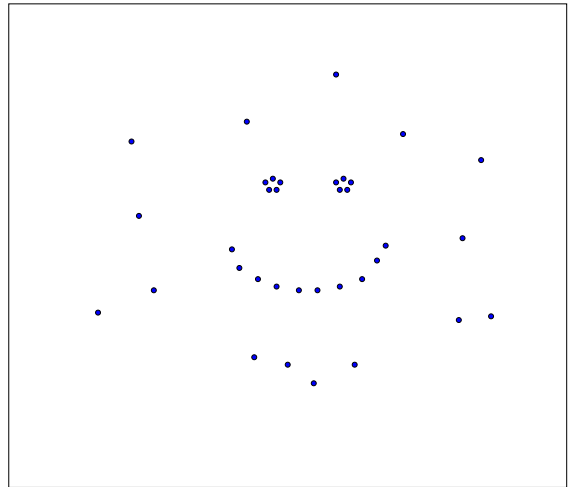
1. An order- $k$  Voronoi diagram partitions the plane according to the  $k$  closest sites. A Voronoi cell in this generalization is the region of points which have the same  $k$  nearest sites. The order-1 Voronoi diagram is nothing more than the standard Voronoi diagram. The order- $(n-1)$  Voronoi diagram is also called the farthest-point Voronoi diagram, because the Voronoi cell of a point  $p_i$  now is the region of points for which  $p_i$  is the farthest site.

(a) Draw (on this sheet) the order-2 Voronoi diagram of the following point set (a).

(b) Draw (on this sheet) the farthest-point Voronoi diagram of the following point set (b).



(a)



(b)

2. Suppose that we are given a subdivision of the plane into  $n$  convex regions. We suspect that this division is a Voronoi diagram, but we do not know the sites. Design and analyze an algorithm that finds a set of  $n$  point sites whose Voronoi diagram is exactly the given subdivision, if such a set exists.
3. The randomized incremental algorithm discussed in class computes the Delaunay triangulation of a set of  $n$  points in  $O(n \log n)$  expected time. Prove that the worst-case running time of the algorithm is  $\Omega(n^2)$ .

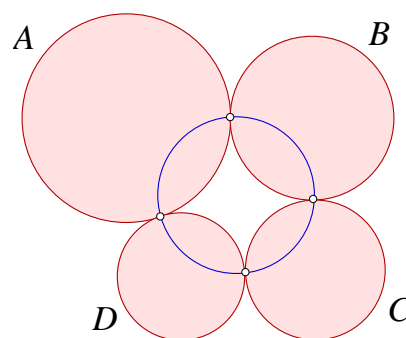
4. Recall that the Delaunay Triangulation (DT) of a set of points in the plane is a maximally connected planar straight line graph. Other interesting and useful planar graphs (not maximal) include Euclidean Minimum Spanning Tree (EMST), Relative Neighborhood Graph (RNG), and Gabriel Graph (GG). (Check your textbook for their definitions). Prove that  $EMST \subseteq RNG \subseteq GG \subseteq DT$ .
5. The *weight* of a triangulation is the sum of the length of all edges of the triangulation. The *minimum weight triangulation* of a set  $S$  of  $n$  points is one whose weight is minimal among all triangulations of  $S$ . The *minmax angle triangulation* of  $S$  is one whose largest angle is minimized.
  - (a) Prove by example that the minimum weight triangulation is not same as the Delaunay triangulation. Show that the ratio of the weight of the Delaunay triangulation to the weight of the minimum weight triangulation is bounded by  $O(n)$ . Show (by a generic example) that your bound is tight.
  - (b) Prove by example that the minmax angle triangulation is not same as the Delaunay triangulation. Prove that the ratio of the largest angle of the Delaunay triangulation to the largest angle of the minmax angle triangulation is bounded from above by 2. Show that this bound is (almost) tight.
6. Let  $P$  be a convex polygon in the plane with  $n$  vertices  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  in counterclockwise order, and let  $z_1, z_2, \dots, z_n$  be arbitrary real numbers. Design and analyze a randomized algorithm to construct the convex hull of the sequence of points  $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_n, y_n, z_n)$  in  $O(n)$  expected time.

## Additional Study Problems

The following problems are not part of the assignment to be graded. Hence, please do not turn in your solutions on them.

1. Recall the plane sweep algorithm for computing Voronoi diagram of a set of  $n$  sites. Give an example where the parabola defined by some site  $p_i$  contributes more than one arc to the beach line. Can you give an example where it contributes  $O(n)$  arcs?
2. Let  $\mathbb{I}^3 = [0, 1] \times [0, 1] \times [0, 1]$  be the unit cube in  $\mathbb{R}^3$  and consider a triangulation  $K$  of  $\mathbb{I}^3$  whose only vertices are the 8 corner points of the cube.
  - (a) Show that every such  $K$  has at most 6 tetrahedra.
  - (b) Show that every such  $K$  has at least 5 tetrahedra.
  - (c) Two triangulations  $K_1$  and  $K_2$  are *isomorphic* if  $\exists$  a bijection  $\beta : \text{Vertices}(K_1) \rightarrow \text{Vertices}(K_2)$  such that  $\text{ConvHull}(T) \in K_1$  iff  $\text{ConvHull}(\beta(T)) \in K_2$  for any  $T \subseteq \text{Vertices}(K_1)$ . Enumerate all pairwise non-isomorphic triangulations of unit cube (with no Steiner points).

3. Let  $A, B, C,$  and  $D$  be four non-overlapping disks such that  $A$  and  $B, B$  and  $C, C$  and  $D,$  and  $D$  and  $A$  are tangent to each other. Prove that these four tangency points are co-circular.



4. Let  $K$  be a triangulation of a set of  $n$  points in the plane. Let  $\ell$  be a line that avoids all points. Prove that  $\ell$  intersects at most  $2n - 4$  edges of  $K$  and that this upper bound is tight for every  $n \geq 3$ .
5. Do exercise 9.16 in the (yellow) textbook.
6. Do exercise 4.15 in the (yellow) textbook.
7. Implement (at least) one of the Delaunay triangulation algorithms discussed in class (edge-flip, randomized incremental, or plane-sweep Voronoi). Run experiments with data sets of various size (say in the range 100-10,000). Does the running time of the implemented algorithm behave as predicted with our analysis? (For this problem you may use libraries for data structures or numerical predicates, but you should implement the main routine yourself.