ASYMPTOTIC LENGTH BOUNDS AND FRACTAL DIMENSION [1]

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**Abstract.**

We provide new results exhibiting relationships between asymptotic length bounds, fractal dimension and generator structure, for various natural classes of uniformly generated piecewise linear curve families.

## Introduction.

Given a uniform sequence of curves in $\Re^2$ with an increasing number of piecewise linear segments, can one find a tight asymptotic bound for the length of a curve in the sequence in terms of the number of curve segments? If so, does this bound have any relationship to the curve's fractal dimension?

The first of these questions arose in the early twentieth century when a man named Richardson was trying to measure the coastline of Great Britain. He observed that the smaller the scale he used, the longer the coastline became. So if one used a sufficiently small scale, then the length of the coastline would appear to be infinite. More recently, Mandelbrot wrote about the similarity apparent in any coastline. He observed that no matter how small the scale, the coastline appeared to have the same shape. Mandelbrot later defined such curves by the name *fractals,* since each has a fractional value corresponding to its dimension.

We deal primarily with expressing the length of piecewise linear curves and curve families in terms of the number of segments in the curve and relating this length bound to the fractal dimension of the curve or curve family. We will concern ourselves with specific types of curves only: uniformly generated, similarly generated, spiraling angle bounded, self-similar and self-avoiding curves, all of which are defined formally and analyzed.

Uniform sequences of curves are specifiable through a "generator" which is just a way of representing the growth of a curve family from one stage to the next. Not only are we trying to find an expression for the length of a curve family in terms of its fractal dimension, we are also trying to find a

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relationship between the length, fractal dimension, and the generator of a curve family.

Another motivation for this project is to find a relation between subdivision schemes and fractals. Subdivision schemes are corner cutting schemes that are used widely for approximating or interpolating a given curve starting from a finite number of points that is usually called a set of control points, or a control polygon. Subdivision schemes construct a sequence of point sequences (which, if linearly connected form a sequence of piecewise linear curves), with increasing number of points. The new points in the sequence are constructed from the old points using a fixed generator (or "mask"). Several basic questions concerning the convergence of the sequence of point sequences generated by a subdivision scheme remain open. In particular, it is not known for what kind of masks and control polygons, the corresponding sequence of point sequences converges to a continuous curve. It is also not known for what kind of masks and control polygons, the corresponding sequence of piecewise linear curves converges to a continuously differentiable curve.

Instead of concerning ourselves with the convergence of a subdivision scheme, the properties of divergent subdivision schemes captured our attention. For example, when will a subdivision scheme diverge to a fractal? Looking at when our generators/masks diverge could lead to learning more about when a subdivision scheme diverges. Another key to the "extent of discontinuity" of a sequence of point sequences generated by a subdivision scheme may lie in the fractal dimension of the corresponding sequence of piecewise linear curves. Looking empirically at how fractal dimension of this sequence varies from mask to mask may provide some clues to answering these questions.

**Organization.** In the first of the following sections we provide some preliminary definitions and background on curve sequences and fractals. The second section is devoted to new results exhibiting relationships between asymptotic length bounds, dimension and generators for various classes of curve families which we define. A Java applet that draws a fractal of the user's choice in one of the above classes is available at http://www.hws.edu/ robitail/Fractal.html. The last section provides numerous open questions and conjectures.

## Preliminaries.

First we need to define the *fractal*, or *Hausdorff dimension* of a curve or curve family.

**Definition** 1.
Consider the metric space $(\Re^m, d)$ where $m$ is a positive integer and $d$ denotes the Euclidean metric. Let $A \subset \Re^m$ which is bounded. Then define the $diam(A)$ to be

$$diam(A) = \sup\{d(x,y) : x, y \in A\}.$$

Let $0 < \epsilon < \infty$ and $0 \leq \rho < \infty$. Let $\mathcal{A}$ denote the set of sequences of subsets $\{A_i \subset A\}$, such that $A = \bigcup_{i=1}^{\infty} A_i$. Then we define

$$\mathcal{M}(A, \rho, \epsilon) = \inf\{\sum (diam(A_i))^{\rho} : \{A_i\} < \epsilon, \text{ for } i = 1, 2, 3, \ldots\}.$$

Let

$$\mathcal{M}(A, \rho) = \sup\{\mathcal{M}(A, \rho, \epsilon) : \epsilon > 0\}.$$

Then for each $\rho \in [0, \infty]$ we have $\mathcal{M}(A, \rho) \in [0, \infty]$. Then the *Hausdorff dimension*, $\delta$, is the unique real number such that

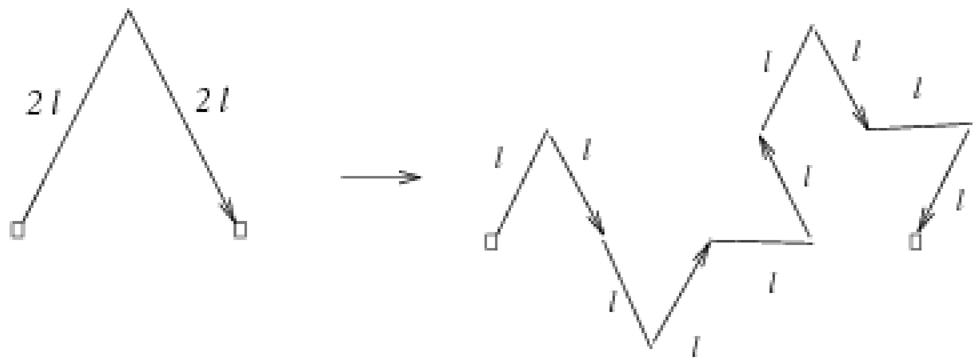
$$\mathcal{M}(A, \rho) = \begin{cases} \infty, & \text{if } \rho < \delta; \\ 0, & \text{if } \rho > \delta \end{cases}$$

[Barnsley, p.201].

Although the Hausdorff dimension is the most useful definition of fractal dimension, it is not usually very easily calculated. So in many cases, the *similarity dimension* is used to estimate the Hausdorff dimension of a curve. In order to define similarity dimension, an *iterated function system (IFS)* must first be defined. For most of the curve families we are concerning ourselves with, the growth of the curve family from one generation to the next can be defined in terms of functions which use previous segments and ratios to generate the next curve. An IFS is such a set of functions.

**Definition** 2.
Given a metric space $S$, an *iterated function system* corresponding to a ratio list $(r_1, r_2, \ldots, r_\sigma)$ is a list $(f_1, f_2, \ldots, f_\sigma)$ where $f_i : S \to S$ is a similarity (i.e. a reflection, a rotation, a dilation or a combination of the three) with ratio $r_i$ [Edgar, p.105].

One example is




Let's call the curve on the left, $C$. Here the ratio list is $(1/2, 1, 2, 1/2, 1/2, 1/2)$ and the function list would be something like this:

- $f_1 =$ dilate($C$) starting from the starting endpoint of $C$
- $f_2 =$ dilate($C$) and reflect($C$) starting at end of curve $f_1$,
- $f_3 =$dilate($C$), reflect($C$), and rotate($C, 60°$) starting at end of curve $f_2$,
- $f_4 =$ dilate($C$) starting at end of curve $f_3$,
- and $f_5 =$ dilate($C$), and rotate($C, -60°$) starting at the end of curve $f_4$.

**Definition 3.**
Given a curve family $C_i$ with an iterated function system realizing a ratio list $(r_1, r_2, \ldots, r_\sigma)$, the *similarity dimension* of the curve family is the value $s_C$ which satisfies the equation

$$1 = {r_1}^{s_C} + {r_2}^{s_C} + \ldots + {r_\sigma}^{s_C}.$$

Whether calculated by using the Hausdorff or similarity dimension, the fractal dimension of a curve is a way to measure how much volume or area the curve takes up in $\Re^2$.

**Definition 4.**
A curve family $C_i$ is said to be *self-similar* if every curve $C_{j+1}$ is obtained from $C_j$ by applying the same IFS to each segment of $C_j$.

**Theorem 5.** (known)

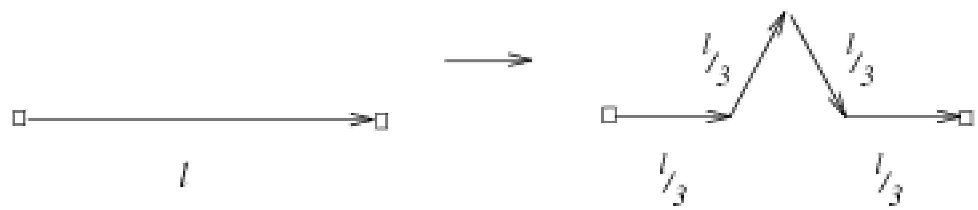
Let $C_i$ be a self-similar piecewise linear curve family which is converging to the fractal $C$ as $i$ tends to $\infty$. Then

$$\delta_C = s_C$$

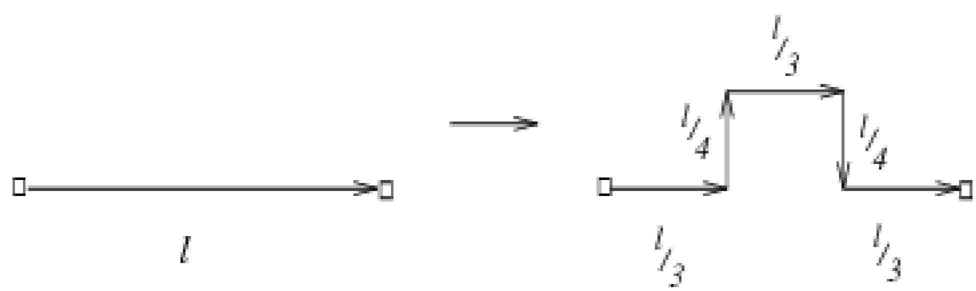
where $\delta_C$ is the Hausdorff dimension of $C$ and $s_C$ is the similarity dimension of $C$.

**Examples.**

The Koch Curve:




The 90° Generator:




### Length, Dimension and Generators.

Before we go on, a few assumptions must be made. The first is that the distance between the two endpts of a curve $C$ is assumed to be one, unless otherwise stated. Also, we use $c_n$ to denote a curve in a curve family $C_i$ containing $n$ segments. It is not necessary that $c_n \in C_i \, \forall \, n \in Z$.

**Definition 6.**
A curve family, $C_i$, is said to be *uniformly generated* if there is some uniform way in which the curve family grows and changes. In other words, there are

some rules, however vague, governing the growth for the curve family from one generation to the next.

**Proposition** 7.
Let $C_i$ be a uniformly generated piecewise linear curve family, converging to the fractal $C$ as $i$ tends to $\infty$ and let $c_n$ be the curve in $C_i$ that has $n$ segments. Then

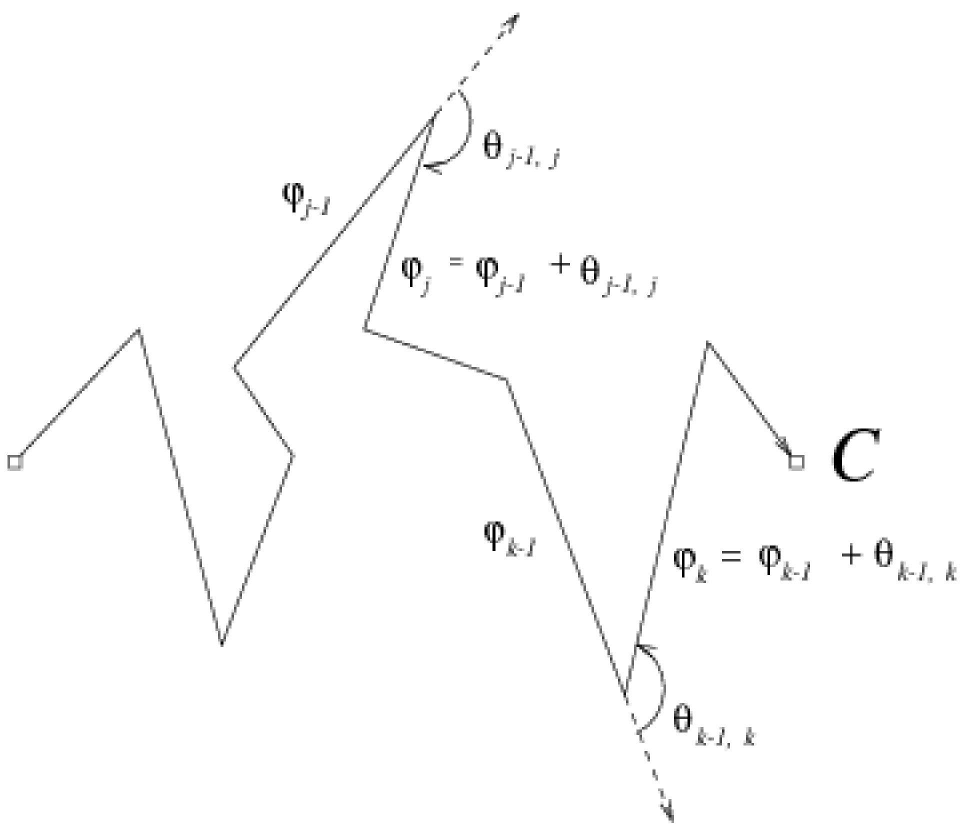
$$|c_n| \leq O(n^{1-1/\delta_C}),$$

where $\delta_C$ is the Hausdorff dimension of $C$.

**Proof.** By definition of the Hausdorff dimension of $C$, it follows that $\delta_C$ is the maximum real value for which $\mu_C =_{def} \lim_{n\to\infty} \sum_i^n |c_n(i, i+1)|^{\delta_C}$ is finite. In order to maximize $|c_n| = \sum_i^n |c_n(i, i+1)|$, assume that for any $1 \leq i, j \leq n$ $|c_n(i, i+1)| = |c_n(j, j+1)|$. Therefore, $|c_n| = \sum_i^n |c_n(i, i+1)| \leq n(\mu_C/n)^{1/\delta_C}$. In other words, $|c_n| \leq O(n^{1-1/\delta_C})$ since $\mu_C$ can be treated as a constant. ♣

**Definition** 8
A curve family $C_i$ is said to be *similarly generated* if for the ratio list $(r_1, r_2, \ldots, r_\sigma)$ of the corresponding iterated function system, there exists an $i$ and $j$ such that $r_i \neq r_j$.

**Explanation.**
For a piecewise linear curve $C$, each segment $j$ in the curve will be assigned a corresponding angle measure, $\phi_j$. For the first segment, $\phi_1$ is just the angle measured from the horizontal to that segment. For any other segment $j$, $\phi_j$ can be found by using $\phi_{j-1}$ and the angle between segment $j-1$ and $j$, call it $\theta$. To find $\theta$, at the endpoint shared by segments $j-1$ and $j$ extend segment $j-1$ out. Then treating this extension as the horizontal, $\theta$ is equal to the angle measured from that extension to segment $j$. So $\phi_j = \phi_{j-1} + \theta$.




**Definition 9**
The ***generator*** for a curve family $C_i$ is the pattern by which one segment in the curve of $C_i$ at stage $j$ generates new segments for the stage $j+1$. For most similarly generated curve families, the generator will be an alternative representation of a curve family's IFS.

**Definition 10**
Given the generator $G$ made up of $n$ segments. Let $\omega = \min_{1 \leq j \leq n} \{|\phi_j|\}$. Then the ***growth factor,*** $\beta_C$ ***of the curve*** $C$ can be expressed as

$$\beta_C = \gcd\{\phi_j - \omega : 1 \leq j \text{ and } |\phi_j| \neq \omega\}.$$

**Definition 11**
An efficient way to represent generators is to use matrices. The growth of a similarly generated curve family $C_i$ from stage $j$ to $j+1$ can be represented

by two generator matrices which will be called segment and length generators. To create either matrix, the rows and columns must be identified by angles. The most useful angles will be those corresponding to $\omega$ and multiples of $\beta_C$. The rows and columns should be identified such that $\angle_{row_i} = \angle_{column_i} \forall i$. The segment generator matrix is the matrix $A$ in the following equation

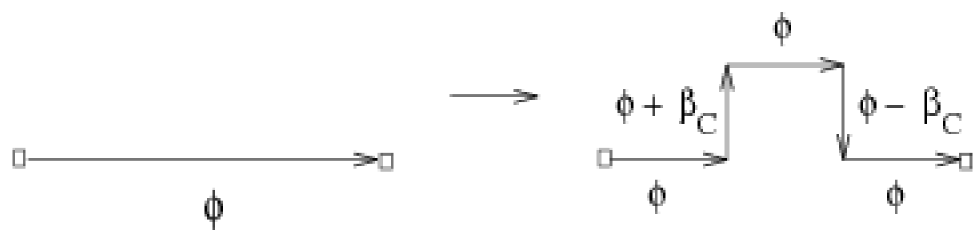
$$s_{j+1} = As_j$$

where $s_{j+1}$ and $s_j$ are column matrices representing the number of segments in stages $j+1$ and $j$, respectively. The length generator matrix is the matrix $B$ in the following equation

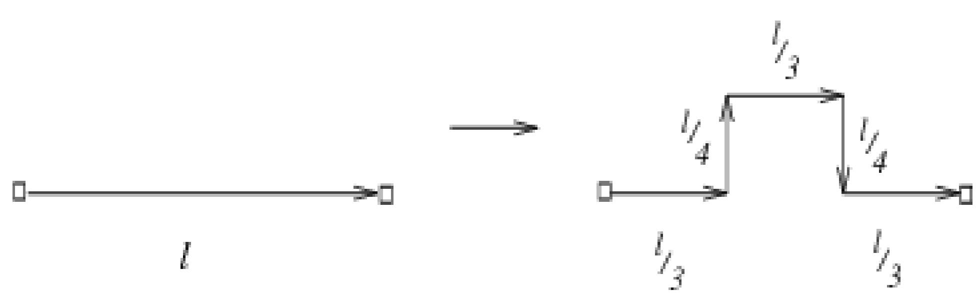
$$l_{j+1} = Bl_j$$

where $l_{j+1}$ and $l_j$ are the column matrices representing the number of segments in the stages $j+1$ and $j$, respectively. It is assumes that the angles identified with the rows of a column matrix are in the same order as the angles identified with the rows of $A$ or $B$.

**Example.** Suppose we have the following generator:
Angles:




Length:



Let $\angle_{row_1} = \angle_{column_1} = |0°|$ , $\angle_{row_2} = \angle_{column_2} = |90°|$ and $\angle_{row_3} = \angle_{column_2} = |180°|$. If we assume all segments in any stage $i$ are of equal length, then we can say that

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1/3 & 1/4 & 0 \\ 1/4 & 1/3 & 1/4 \\ 0 & 1/4 & 1/3 \end{bmatrix}.$$

Even though a segment with $\phi = |180°|$ also produces a segment with $\phi = |270°|$, this growth can only be represented if we increase the size of our generator matrices.

**Definition 12**
A curve family $C_i$ is said to be *equi-similarly generated* if for the ratio list $(r_1, r_2, \ldots, r_\sigma)$ of the corresponding iterated function system, for any $1 \leq i, j \leq \sigma, r_i = r_j$. In other words, all segments in a given stage have the same length.

**Definition 13**
Given any piecewise linear curve $C$ made up of $n$ segments, $C$ is said to be *spiraling angle bounded by* $\alpha_C$ if

$$\alpha_C = \max_{1 \leq j \leq n} \{|\phi_j|\}.$$

**Definition 14**
The curve family $C_i$, which is converging to the fractal $C$ as $i$ tends to $\infty$, is *spiraling angle bounded* if for any two curves $c_m$ and $c_n$ where $m$ and $n$ are sufficiently large and $m < n$, $\alpha_{c_m} = \alpha_{c_n}$; i.e. $c_m$ and $c_n$ have the same spiraling angle bound. Furthermore, it can be said that the spiraling angle bound, $\alpha_C$, is $\alpha_C = \alpha_{c_m} = \alpha_{c_n}$.

**Proposition 15.**
Given a piecewise linear curve $C$ with a spiraling angle bound $\alpha_C < 90°$, the length of $C$ can be expressed as

$$|C| = kd$$

where $k$ is some constant and $d$ is the distance between the endpoints of $C$.

**Proof.** Assume that $C$ has $n$ segments. The length, $d$, can be broken up into $n$ segments denoted by $d_{i,i+1}$ where $1 \leq i < n$ and $d_{i,i+1}$ is the horizontal distance between the two endpoints of segment $i$. Then for each $d_{i,i+1}$, $|C(i,i+1)| \leq \frac{d_{i,i+1}}{\cos \alpha_C}$. Altogether,

$$|C| = \frac{1}{\cos \alpha_C} \sum_{i=0}^{n-1} d_{i,i+1} = \frac{1}{\cos \alpha_C} d.$$

Since $\alpha_C < 90°$, $\frac{1}{\cos \alpha_C}$ is some constant, k. Therefore $|C| = kd$. ♠

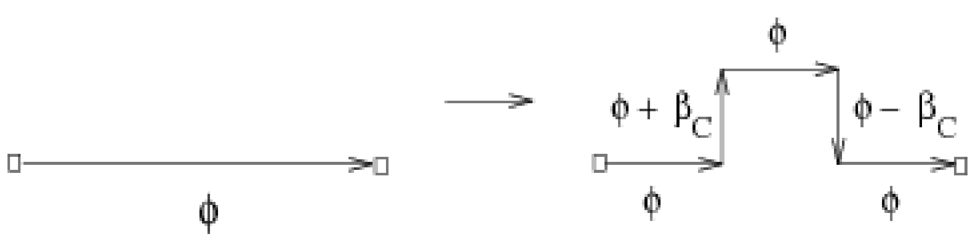
**Remark.** Since $|C| \leq kd$ where $k$ is some constant, the curve $C$ is said to be *k-rectifiable* . However even if a uniformly generated sequence of $k$-rectifiable curves converges to a continuous curve $C$, the length of $C$ may not be equal to the limit of the lengths of the sequence.

**Proposition** 16.
Given a piecewise linear curve $C$ with a spiraling angle bound $\alpha = 90°$, the length of $C$ can be expressed as

$$|C| = O(\log n),$$

where $n$ is the number of segments in $C$.

**Proof.** Consider the following generator:




Since $\alpha_C = 90°$ and $\beta_C = 90°$, growth only takes place on segments which have angle measure 0°. So the segments with an angle measure $|90°|$ do not change from one stage to the next stage. This generator can be represented in matrix form as:

$$\begin{bmatrix} \#h_j \\ \#v_j \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \#h_{j-1} \\ \#v_{j-1} \end{bmatrix}$$

where $\#h_j$ and $\#v_j$ are, respectively, the number of new horizontal and vertical segments at stage $j$ $\forall j$ and

$$\begin{bmatrix} |h_j| \\ |v_j| \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} |h_{j-1}| \\ |v_{j-1}| \end{bmatrix}$$

where $|h_j|$ and $|v_j|$ represent the lengths of new horizontal and vertical segments, respectively, in stage $j$ $\forall j$. By combining the first matrix equation with the fact that for every 3 new horizontal segments in stage $j-1$ there are 2 unchanging vertical segments, it is easy to see that the total number of segments at stage $j$ is

$$n = 3(\#h_{j-1}) + 2(\#h_{j-1}) + \frac{2}{3}(\#h_{j-1}) = \frac{17}{3}(\#h_{j-1}).$$

Since $\#h_{j-1} = 3\#h_{j-2}$ and $\#h_0 = 1$,

$$n = \frac{17}{3}(3^{j-2}) = 17(3^{j-3}).$$

Then we can express $j$ as $j = 3 + \log_3 n - \log_3 17$. To find $|C|$, we can say that $|C| =$ *vertical length of* $C$ + *horizontal length of* $C$. Since growth only occurs on horizontal segments, the horizontal length of the curve at any stage is $d$. Also, since any new vertical growth is $\frac{2}{3}d$, the vertical length is $\frac{2}{3}jd$. Thus

$$|C| = \frac{2}{3}jd + d = \frac{2}{3}(3 + \log_3 n - \log_3 17)d + d.$$

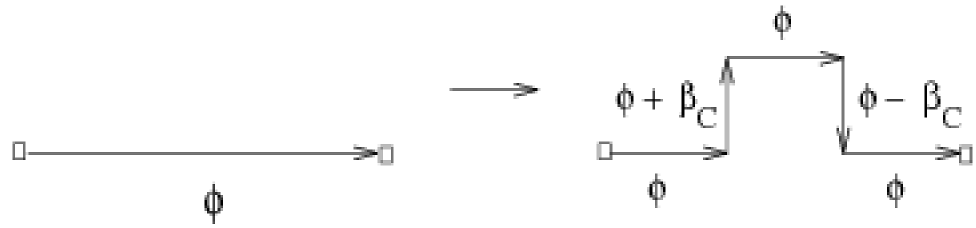
Therefore

$$|C| \leq O(\log n).$$

$\diamondsuit$

**Remark.** From Proposition 7, $|C| \leq O(n^{1-1/\delta_C})$. Since $\log n = n^{\frac{\log\log n}{\log n}}$ and $\lim\limits_{n\to\infty} \frac{\log\log n}{\log n} \to 0$, the Hausdorff dimension of such a curve would be $\delta_C = 1$. Therefore, such a curve $C$ with a spiraling angle bound, $\alpha_C = 90°$ would be a non-rectifiable curve with fractal dimension 1.

**Proposition** 17.
Given the generator in the following figure:




Let $A$ be the segment generator matrix for a equi-similar and spiraling angle bounded curve family $C_i$ which is converging to the fractal $C$ as $i$ tends to $\infty$. Assume that the segment generator has a growth factor of $\beta_C = 90°$ and that the spiraling angle bound, $\alpha_C$, is a multiple of 90°. Then any entry $A_{ij}$ where $i$ is the row number and $j$ is the column number can be expressed as

$$A_{ij} = \begin{cases} 3, & \text{if } \theta_i = \theta_j; \\ 2, & \text{if } \theta_i = \theta_2 \text{ and } \theta_j = \theta_1; \\ 1, & \text{if } \theta_j = \theta_i \pm 90° \text{ and } \theta_j \neq \alpha_C; \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta_i$ and $\theta_j$ are the angle measures corresponding to row $i$ and column $j$, respectively, and $\theta_1 = 0°$.

**Proof.** Consider the growth of a segment with an angle measure of $\phi$ where $|\phi| \leq \alpha_C$.
Case 1: $\phi \neq \alpha_C$.

Since $\phi \neq \alpha_C$ this segment will generate 3 segments of angle measure $\phi$, 1 segment of angle measure $\phi - 90°$ and 1 segment of angle measure $\phi + 90°$. So if $\phi = \theta_i$, then $A_{ii} = 3$, $A_{i-1,i} = 1$ and $A_{i+1,i} = 1$. In the case where $\phi = 0°(= \theta_1)$, $|\phi - 90°| = |\phi + 90°|$. Therefore $A_{2,1} = 2$.
Case 2: $\phi = \alpha_C$.

Since $\phi = \alpha_C$, no growth can occur on this segment and this segment will become 3 segments with the same angle measure $\phi$. Therefore if $\phi = \theta_i$, $A_{i,i} = 3$ and $A_{i-1,i} = 0$.

**Lemma** 18
Let $C_i$ be a equi-similarly generated piecewise linear curve family that has a spiraling angle bound of $\alpha_C \geq 90°$ and is converging to the fractal $C$ as $i$ tends to $\infty$. Let $c_n$ be the curve in $C_i$ with $n$ segments. Assume the segment

generator matrix for $C_i$ has a unique largest real eigenvalue, call it $\lambda_{max}$. Then

$$n \leq O(\lambda_{max}^k)$$

where $k$ is the stage in which $c_n$ is produced.

**Proof.** The number of segments at any stage, $k$, can be found by using the segment generator matrix, $A$, and a vector, $X_0$, corresponding to the initial curve $c_1$ of the curve family $C_i$. The number of segments, $s(k)$ can be calculated by using the matrix formula $s(k) = \sum_{j=1}^{b} [A^k X_0]_{j,1}$ where $b = \beta_C/\alpha_C$. By finding the eigenvalues and eigenvectors of $A$, the above equation becomes $s(k) = \sum_{j=1}^{b} [PD^k P^{-1} X_0]_{j,1}$ where $D$ is the diagonalized eigenvalue matrix, $P$ is the corresponding eigenvector matrix and $P^{-1}$ is the inverse of $P$†. By solving this equation, one finds that $s(k)$ can be written as a linear combination of $A$'s eigenvalues to the power of $k$ where the coefficients of each term are independent of $k$. As $k$ gets large, $\lambda_{max}^k$ becomes the dominant term. In other words, $s(k) \leq O(\lambda_{max}^k)$. ♡

† – In cases where $P$ is singular, one would use the equation $s(k) = \sum_{j=1}^{b} [A^k X_0]_{j,1}$ where $X_0$ can be written as a combination of the eigenvalues of $A$.

**Proposition** 19.
Let $C_i$ be a equi-similarly generated piecewise linear curve family that has a spiraling angle bound of $\alpha_C \geq 90°$ and is converging to the fractal $C$ as $i$ tends to $\infty$. Let $c_n$ be the curve in $C_i$ that has $n$ segments. Then

$$|c_n| \leq O(n^{1-\log_{\lambda_{max}} R}).$$

**Proof.** Assume all the segments in $c_n$ are of equal length and are of length $1/R$ of the length of the segments in the previous generation. Consider the generator matrices which produce the curve family $C$. The length of $c_n$ can be represented as $|c_n| = l(j)s(j)$ where $l(j)$ is the length of any segment in the curve at stage $j$ and $s(j)$ is total number of segments in the curve at stage $j$. Since the length of a segment at any stage $j$ is $1/R$ of the length of a segment at stage $j-1$, clearly $l(j) = (1/R)^j$. By Lemma 18, $s(j) \leq O(\lambda_{max}^j)$ where $\lambda_{max}$ is the largest eigenvalue of the segment generator matrix $A$. Then

$i \leq O(\log_{\lambda_{max}} s(j))$. So the equation $|c_n| = l(j)s(j)$ becomes

$$|c_n| \leq O((1/R)^{\log_{\lambda_{max}} s(j)} s(j)).$$

Since $s(j) = n$ and $a^{\log b} = b^{\log a}$,

$$|c_n| \leq O(n^{1-\log_{\lambda_{max}} R}).$$

♠

**Lemma 20.** ( this really could be false – must check it further )
Let $C_i$ be a equi-similarly generated piecewise linear curve family that has a spiraling angle bound of $\alpha_C \geq 90°$ and is converging the fractal $C$ as $i$ tends to $\infty$. Assume $C_i$ has a growth factor $\beta_C = 90°$. Then

$$\lambda_{max} \leq R + A_{1,2}\left(2 + 2\cos\frac{\pi\beta_C}{\alpha_C}\right)^{1/2}$$

where R is the ratio between the length of a segment in stage $j$ and the length of a segment in stage $j+1$.

**Proof. (Empirical Sketch)** In order to get a bound on $\lambda_{max}$ in terms of the spiraling angle bound $\alpha_C$, I used Maple to first find the eigenvalues for the generator defined in Proposition 17. By looking at the largest eigenvalues of this generator at different spiraling angle bounds, I hoped to find a pattern in these eigenvalues from one spiraling angle bound to the next. When I asked Maple for the eigenvalues for the generator with a spiraling angle bound of $\alpha_C = 810°$, it returned:

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bytes used=1000268, alloc=917336, time=1.28
bytes used=2000572, alloc=1179432, time=2.72

$3 + 3^{1/2}$ , $3 - 3^{1/2}$ , $3 + (2 + 2\cos(1/9\ \mathrm{Pi}))^{1/2}$ , $3 - (2 + 2\cos(1/9\ \mathrm{Pi}))^{1/2}$ ,

$3 + (2 - \cos(1/9\ \mathrm{Pi}) - \%1)^{1/2}$ , $3 - (2 - \cos(1/9\ \mathrm{Pi}) - \%1)^{1/2}$ ,

$3 + (2 - \cos(1/9\ \mathrm{Pi}) + \%1)^{1/2}$ , $3 - (2 - \cos(1/9\ \mathrm{Pi}) + \%1)^{1/2}$ , 3, 3

$\%1 := 3^{1/2} \sin(1/9\ \mathrm{Pi})$

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which was surprising. I did not expect to see sines and cosines in the eigenvalues. The results up until this point seemed fairly random, but the prospect of them steming from some trigonometric equation was intriguing. So I decided to see if all the results I had obtained thus far agreed with this new found equation $\lambda_{max} = 3 + (2 + 2\cos\frac{\pi\beta_C}{\alpha_C})^{1/2}$. I noticed that for smaller $\alpha_C$ this equation definitely held. As $\alpha_C$ increases though, this equation is slightly larger than the corresponding $\lambda_{max}$. I concluded that for this specific generator, $\lambda_{max} \leq 3 + (2 + 2\cos\frac{\pi\beta_C}{\alpha_C})^{1/2}$. Then I looked to see if I could find a similar type of equation for other generators. To my surprise, by generalizing the about result, I had a bound for the eigenvalues of the other generators as well. Therefore, I have come to the conclusion that

$$\lambda_{max} \leq R + A_{1,2}\Big(2 + 2\cos\frac{\pi\beta_C}{\alpha_C}\Big)^{1/2}.$$

**Theorem** 21.
Let $C_i$ be a equi-similar piecewise linear curve family that has a spiraling angle bound of $\alpha_C \leq 90°$ and is converging to the fractal $C$ as $i$ tends to $\infty$. Let $c_n$ be the curve in $C_i$ with $n$ segments. Then

$$|c_n| \leq O(n^{1-\xi}),$$

where $d$ is the distance between the two endpoints of $C$ and

$$\xi = \frac{\log R}{\log\Big(R + A_{1,2}\left(2 + 2\cos\frac{\pi\beta_C}{\alpha_C}\right)^{1/2}\Big)}.$$

**Proof.** From Proposition 17, we have $|c_n| \leq O(n^{1-\log_{\lambda_{max}} R})$. By Lemma 20, $\log_{\lambda_{max}} R \leq \xi$. Therefore,

$$|c_n| \leq O(n^{1-\xi}).$$

$\diamondsuit$

**Lemma** 22.
Let $C_i$ be a self-similarly generated piecewise linear curve family that is converging to the fractal $C$ as $i$ tends towards $\infty$. Let $(1/\nu_1, 1/\nu_2, \ldots, 1/\nu_\sigma)$ be the ratio list realized by the corresponding IFS. Assume that for the initial

curve $C_0$, $|C_0| = 1$. Let $c_n$ be the curve in $C_i$ containing n segments. Then the average length of a segment at stage j is

$$l(j) = \frac{1}{\sigma^j}\Big(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\Big)^j.$$

**Proof by Induction.**

Base Case: $j = 1$. Let $C_1 \in C_i$ be the curve which represents the curve family at stage $j = 1$. Since $C_0$ is reproduced $\sigma$ times in the curve $C_1$ by similarities with ratios $(1/\nu_1, 1/\nu_2, \ldots, 1/\nu_\sigma)$, the average length of a segment in $C_1$ is

$$l(1) = \frac{1}{\sigma}\Big(\frac{1}{\nu_1}|C_0| + \frac{1}{\nu_2}|C_0| + \ldots + \frac{1}{\nu_\sigma}|C_0|\Big).$$

Since $|C_0| = 1$,

$$l(1) = \frac{1}{\sigma}\Big(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\Big) = \frac{1}{\sigma^1}\Big(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\Big)^1.$$

Therefore the statement $l(j) = \frac{1}{\sigma^j}(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})^j$ holds when $j = 1$.

Inductive Case: Assume the statement holds for $j = k$ for some $k \geq 1$. We want to show that the statement holds for $j = k + 1$, i.e. $l(k + 1) = \frac{1}{\sigma^{k+1}}(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})^{k+1}$. Consider $l(k + 1)$. Each segment of curve $C_k$ is reproduced $\sigma$ times in the curve $C_{k+1}$ by similiarities with ratios $(1/\nu_1, 1/\nu_2, \ldots, 1/\nu_\sigma)$. The average length of one of these $\sigma$ new segments can be expressed as

$$l(k+1) = \frac{1}{\sigma}\Big(\frac{1}{\nu_1}l(k) + \frac{1}{\nu_2}l(k) + \ldots + \frac{1}{\nu_\sigma}l(k)\Big),$$

i.e.

$$l(k+1) = \frac{1}{\sigma}\Big(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\Big)l(k).$$

Since $l(k) = \frac{1}{\sigma^k}(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})^k$, $l(k + 1)$ can be rewritten as

$$l(k+1) = \frac{1}{\sigma^{k+1}}\Big(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\Big)^{k+1}.$$

Therefore the statement $l(j) = \frac{1}{\sigma^j}(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})^j$ holds for $j = k + 1$ if it holds for $j = k$ $\forall\, k \geq 1$.

Since the statement, $l(j) = \frac{1}{\sigma^j}(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})^j$, holds for $j = 1$ and if it holds for value, then it holds for the next, by the principle of induction, the statement holds for all $j \geq 1$.

**Lemma 23.**
Let $C_i$ be a self-similarly generated piecewise linear curve family that is converging to the fractal $C$ as $i$ tends towards $\infty$. Consider the ratio list $(1/\nu_1, 1/\nu_2, \ldots, 1/\nu_\sigma)$. Then

$$\log_\sigma \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma} \right) \leq 1 - 1/s_C$$

where $s_C$ is the similarity dimension of the fractal $C$.

**Proof.** Let $\gamma = (\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})$. In order to maximize the value of $\gamma$, assume that for all $1 \leq \kappa \leq \sigma, \nu_\kappa = \nu$ for some $\nu > 0$. Then $\gamma$ becomes $\gamma = \sigma/\nu$ for any positive $\sigma$. Consider the similarity dimension, $s_C$ of the fractal $C$. By definition, $s_C$ is the solution to the equation $1 = \sigma(1/\nu)^{s_C}$. Then $s_C = \log_{1/\nu} 1/\sigma$, i.e. $s_C = \frac{1}{\log_\sigma \nu}$. Also consider $\log_\sigma \gamma$. Since $\gamma = \sigma/\nu$, $\log_\sigma \gamma = \log_\sigma \sigma/\nu = 1 - \log_\sigma \nu$, by a property of logarithms. Since $s_C = \frac{1}{\log_\sigma \nu}$,

$$\log_\sigma \gamma = 1 - 1/s_C.$$

Therefore, in general,

$$\log_\sigma \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma} \right) \leq 1 - 1/s_C.$$

♠

Although Proposition 7 gives a bound on the lengths of similarly generated curves in terms of the Hausdorff dimension, we would prefer to have a bound in terms of the similarity dimension because it is easier to obtain.

**Proposition 24.**
Let $C_i$ be a similarly generated piecewise linear curve family that is converging to the fractal $C$ as $i$ tends towards $\infty$. Let $c_n$ be the curve in $C_i$ containing n segments. Then

$$|c_n| < O(n^{1-1/s_C}),$$

where $s_C$ is the similarity dimension of $C$.

**Proof.** Without loss of generality, since we are looking for a bound on $c_n$, we can assume the curve family is *self-similar, otherwise the bound is not tight.* The length of $c_n$ can be represented as $|c_n| = l(j)s(j)$ where $l(j)$ is a function giving the length of a segment at stage j and $s(j)$ is the number of segments in the curve $c_n$. If every segment in stage j becomes m segments in stage j+1, then $s(j) = m^j$. Since $s(j) = n$, the stage j can be written as $j = \log_m n$. By Lemma 18, $l(j) = \frac{1}{\sigma^j}(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma})^j$. So the equation $|c_n| = l(i)s(i)$ becomes

$$|c_n| = \frac{1}{\sigma^j}\left(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\right)^j n.$$

Substituting in for $j$ and simplifying gives

$$|c_n| = \left(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\right)^{\log_\sigma n}.$$

Since $a^{\log b} = b^{\log a}$,

$$|c_n| = n^{\log_\sigma \left(\frac{1}{\nu_1} + \frac{1}{\nu_2} + \ldots + \frac{1}{\nu_\sigma}\right)}.$$

Using the result from Lemma 23,

$$|c_n| \le O(n^{1-1/sc}).$$

♣

**Definition 25**

A curve is said to be *k-self-avoiding* for a given constant $k$ if for any two points of the curve, say $i$ and $j$, an ellipse with foci $i$ and $j$ can be made using a string of length $d_{ij}$ where $d_{ij}$ is the distance between $i$ and $j$, such that this ellipse contains the whole curve between the points $i$ and $j$.

**Theorem 26.**

Let $P$ be a plane-filling curve. Then $P$ cannot be $k$-self-avoiding for any $k$.

**Proof.** Given $k > 1$. Assume $P$ is a plane-filling curve that is $k$-self-avoiding. If at any stage in the construction of $P$, $P$ is self-intersecting or self-contacting, then clearly $P$ is not $k$-self-avoiding. Otherwise, we can find, for every $M > 0$ a stage, call it $i$, in the construction of $P$ where $P_i$ goes through everypoint in the $M$-lattice (i.e. a plane grid where the distance between adjacent points is $1/M$). Since $P$ sis $k$-self-avoiding, depending on $k$, there exists a number $N(k)$ which represents the maximum number of lattice points on a portion of $P$ whose endpoints are adjacent on the lattice. Notice that $N(k)$ is independent of the lattice. We can find an $M > 0$ such that for any curve $P$ which goes through all the points of an $M$-lattice, there must exist two adjacent points, call them $\gamma$ and $\zeta$ such that the curve between $\gamma$ and $\zeta$ must contain strictly more than $N(k)$ points. Then the curve between $\gamma$ and $\zeta$ escapes the ellipse with foci $\gamma$ and $\zeta$ and with string length $kd_{\gamma\zeta}$, where $d_{\gamma\zeta}$ is the distance between the points $\gamma$ and $\zeta$. Therefore $P$ is not $k$-self-avoiding. ♡

**Proposition** 27.
Let $k \leq \sqrt{2}$ and $C_i$ be a $k$-self-avoiding curve family converging to the fractal $C$ as $i$ tends to $\infty$. If $c_n$ is the curve in $C_i$ with $n$ segments, then the length of $c_n$ can be expressed as

$$|c_n| \leq O(n^{\log_2 k}).$$

**Proof.** Assume that the curve $c_n$ appears in the $j$th stage of construction of the fractal $C$. In order to maximize the length of the curve $c_n$, assume that the length of the curve increases by a factor of $k$ from one stage to the next. So the length of the curve $c_n$ can be expressed as $|c_n| \leq k^j d$. Also, assume that any segment in one stage will generate two segments in the next stage. Then we can say that $n = 2^j$, or $j = \log_2 n$. By substitution and a property of logarithms, we can write the length of $c_n$ as

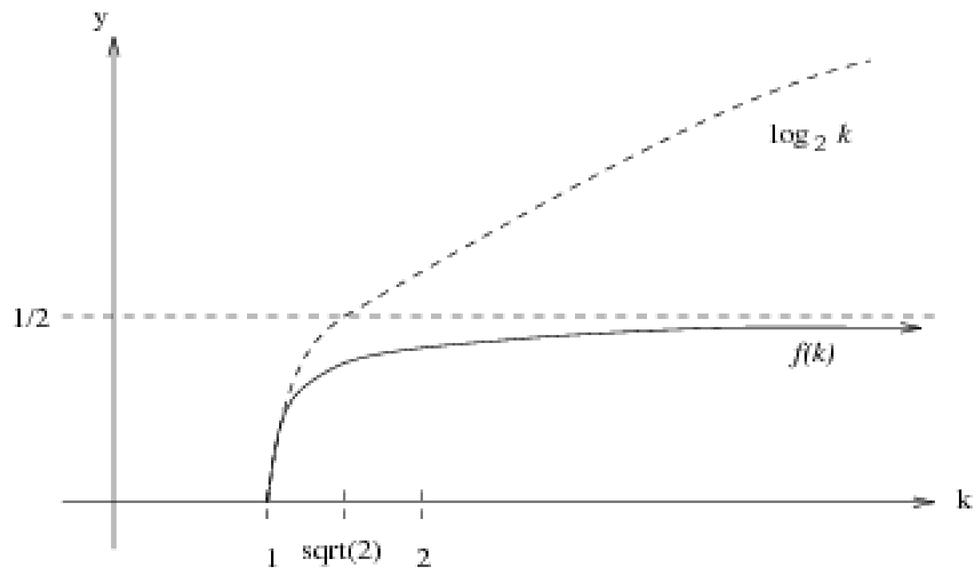
$$|c_n| \leq O(k^{\log_2 n}) \leq O(n^{\log_2 k}).$$

♠

**Conjecture** 28.
Let $c_n$ be a $k$-self-avoiding curve with $n$ segments. Then the length of $c_n$ can be expressed as

$$|c_n| \leq O(n^{f(k)})$$

where $f(k)$ is the function graphed below.




**Open Questions**

- When does the sequence of point sequences generated by a subdivision scheme or uniform generation scheme converge to continuous curve (modulo finitely many discontinuities)? (This is a DIFFICULT question).

- (This probably just requires some basic differential geometry and topology) If convergence does take place as above, does the total-length sequence of the corresponding sequence of piecewise linear curves converge to a limit? In other words, is the sequence of piecewise linear curves $k$-rectifiable for some $k$? (Notice, however, that the limit of the total lengths may not be the same as the length of the limit curve– consider the staircase with increasing number of steps and 45 degree gradient, which converges to the diagonal of the corresponding square and whose length is 1, but the limit of the lengths of the curve sequence is $\sqrt{2}$.) How about viceversa: if the sequence of total lengths converges to a limit, then does the sequence of point sequences converge to a continuous curve (or atmost with finitely many discontinuities)?

- (This probably also just requires some basic differential geometry and topology) Is it the case that the sequence of point sequences generated

by a subdivision scheme converges to a continuous curve
if and only if
the corresponding sequence of piecewise linear curves converges to a continuous curve (modulo finitely many discontinuities)
if and only if
both converge to a $C^1$ curve (modulo finitely many points where the curve is not $C^1$).

- (This is probably also fairly easy) Due to time constraints, the relation between a generator and a mask was not found. The question of finding this relation is still open. It seems rather obvious that a relation should exist; it's just a matter of having the time and patience to find it.

- What is the break point between a subdivision scheme which converges (or diverges) and a subdivision scheme which diverges to a fractal? In particular, find a sequence of masks/generators $G_1 \ldots G_{b_1} \ldots G_{b_2}$ so that

  - for $G_1 \ldots G_{b_1}$, the corresponding sequence of point sequences converges to continuous curves,
  - for $G_{b_1} \ldots G_{b_2}$, the corresponding sequence of point sequences does not converge to anything close to a continuous curve (the length sequences do not converge to a limit, and the sequence is not rectifiable), but the fractal dimension of the sequence is still 1,
  - and finally for each of the remaining generators, we obtain a sequence representing a fractal of dimension strictly greater than 1.

- For both of the latter cases above, what is the relationship between the mask, the asymptotic length (of a divergent length sequence) and the fractal dimension?

- What if there isn't a unique maximum eigenvalue for a generator matrix? How does one go about finding the length of a curve? Also, what would be the relation between such a generator and fractal dimension? What about the geometry of such a generator?

- Why is $\lambda_{max} \leq R + A_{1,2}(2 + 2\cos\frac{\pi\beta_C}{\alpha_C})^{1/2}$ when $\lambda_{max}$ is unique?

- What is the geometric interpretation of this result? In other words, is there a relationship between unique largest eigenvalues and the geometry of a generator?

- One more parameter to look at is the sequence of areas enclosed by a uniformly generated sequence of curves.

- One should look at all the questions mentioned here and studied in this write-up for higher dimensional fractals, say areas of fractal surfaces, volumes enclosed by fractal surfaces etc..

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