

Homework assignment 4

1. The following facts are known about decision Problems U, V, W, X, Y and Z are all in the described below. For each decision problem, classify its complexity as precisely as you can (i.e, give the best lower and upper bounds on its complexity that you can manage). Justify your answer.
 - a. Problem U is the fire-station location problem. The input is a graph $G = (V, E)$, whose vertices are housing lots with positive weights on the edges representing road distances between them, and two numbers k and w . The problem is to find out if one can pick at most k lots that can serve as fire-stations, with the property that every lot is at a total road distance of at most w from atleast 1 fire-station.
 - b. Problem V is the following special type of Boolean satisfiability problem that occurs in logic programming: The input is a Boolean propositional formula F which is a 2-CNF formula is a Boolean formula in conjunctive normal form, but each of whose clauses is a conjunction of exactly 2 literals. The problem is to decide if F is satisfiable.
 - c. There is a function that reduces problem U to problem W . I.e, $U \leq_m W$. The function takes n^2 steps to compute, for an input of size n .
 - d. There is a function that reduces problem W to problem U . I.e, $W \leq_m U$. The computation of the function takes n^2 steps for an input of size n .
 - e. There is a function that reduces problem W to problem X . The computation of the function takes n^3 steps for an input of size n .
 - f. There is a function that reduces problem Y to problem V . The computation of the function takes $n \log n$ steps for an input of size n .
 - g. There is a function that reduces problem V to problem Z . The computations of the function takes n steps for an input of size n .
2. In the proofs that the *TIME* hierarchy is proper (I.e, the proof that $D(N)TIME(f(n))$ is strictly contained in $D(N)TIME(g(n))$, provided $\lim_{n \rightarrow \infty} f(n) \log(f(n))/g(n) = 0$), we missed out an important point. This question requires you to rectify this.

Recall how all the diagonalization proofs went. For this example, we took an $f(n)$ -time computable enumeration of all $O(f(n))$ -time bounded machines M (machines that clocked and are forced to halt and reject when their time requirement exceeds $k \cdot f(n)$ for some k ; and are untouched otherwise). Then we constructed a machine M^* that recognizes a “diagonal” set that differs from every set recognized by the machines M in the above

enumeration. The machine M^* simply simulated M on input $\langle M \rangle$, i.e, the enumeration index of M , and gave the opposite answer as M .

By doing this, we claimed that both

- (i) M^* was $g(n)$ -space bounded and that
- (ii) M^* 's set differed from every set recognized by the machines M in the enumeration.

The latter is clearly true. The former is not. For example, what if for some small value of n , $g(n) < f(n) \log(f(n))$, although $g(n)$ is asymptotically greater than $f(n) \log(f(n))$? One way to fix (ii) is to explicitly bound M^* 's time by $g(n)$, and make it halt and reject if it exceeds this space. But then (i) is no longer true. Why? Can you make a minimal change to M^* to make both (i) and (ii) true? Justify your answer. Answers to these questions should be specific to the issues discussed here. Please do not simply regurgitate a full, correct proof from some textbook without specifically answering the questions above.

3. (i) Is the complexity class P closed under union? intersection? complement? Why?
 - (ii) Is the class NP closed under union? intersection?
 - (iii) It is not known whether NP is closed under complementation. Why does the proof that $r.e \neq co - r.e$ not go through in this case?
 - (iv) The class co-NP consists of languages whose complements are in NP. Show that if a set is NP-complete, then its complement is co-NP-complete. (The definition of co-NP-complete is obtained by replacing NP by co-NP everywhere in the definition of NP-complete).
 - (v) Show that if an NP-complete set belonged to co-NP, then NP = co-NP.
 - (vi) Can you push through the proof that $r.e \cap co - r.e = decidable$ to show that $NP \cap co-NP = P$?
 - (vii) Is it possible that $NP \neq co-NP$, but $P = NP$? How about viceversa?

4. 9.10; 9.18; 9.19; 9.20; ($(N)EXPTIME$ is the complexity class $\bigcup_{k \in \mathbb{N}} (N)DTIME(2^{n^k})$).

5. (Bonus – space complexity) 8.21 (L is the complexity class $DSPACE(\log n)$).