Solution of Homework 3

1. (a) Let $X$ be a set in the class r.e. Let $M$ be a machine that recognizes $X$. Let $f(x) = (M, x)$. Then $x \in X \iff f(x) \in \overline{\text{A}_{\text{pseudocode}}}$. 

(b) The set $\overline{\text{A}_{\text{pseudocode}}}$ is $\leq^\text{rec}_m$-complete for the class co-r.e. Let $X$ be a set in the class co-r.e., i.e. $\overline{X}$ is a set in r.e. Let $\overline{M}$ be a machine that recognizes $\overline{X}$. Let $f(x) = (\overline{M}, x)$. Then $x \in X \iff x \notin \overline{X} \iff (\overline{M}, x) \notin \overline{\text{A}_{\text{pseudocode}}}$. 

(c) This set $A$ cannot be decidable. If it were decidable then there would exist an algorithm for deciding $\overline{\text{A}_{\text{pseudocode}}}$ ($x \in \overline{\text{A}_{\text{pseudocode}}} \iff f(x) \in A$ for some recursive function $f(x)$ and $f(x) \in A$ can be checked in finite time). But we know that $\overline{\text{A}_{\text{pseudocode}}} \notin \text{rec}$, contradiction.

2. (a) Let $X$ be a set in the class r.e. Take $A = X \in \text{r.e.}, B = \Sigma^* \in \text{co-r.e.}$ then $X = A \cap B$. Therefore $X \in \text{Diff}$ and r.e. $\subset \text{Diff}$. Similarly co-r.e. $\subset \text{Diff}$ so r.e. $\cup$ co-r.e. $\subset \text{Diff}$.

(b) Let $D = \{(x, y) : x \in \text{A}_{\text{pseudocode}} \land y \notin \text{A}_{\text{pseudocode}}\}$. Let $X \in \text{Diff}, X = A \cap B, A \in \text{r.e.}, B \in \text{co-r.e.}$ Let $f(x) = (f_A(x), f_B(x))$ where $f_A(x) \in A_{\text{pseudocode}} \iff x \in A$ and $f_B(x) \notin A_{\text{pseudocode}} \iff x \in B$. Then $f(x) = (f_A(x), f_B(x)) \in D \iff f_A(x) \in A_{\text{pseudocode}} \land f_B(x) \notin A_{\text{pseudocode}} \iff x \in A \land x \in B \iff x \in A \cap B = X$. Therefore $D$ is $\leq^\text{rec}_m$-complete for Diff.

3. (a) 6.13 For any $x \in \Sigma^*$ we know that $K(x) \leq |x| + c$ for some constant $c$. So we can simply try all pairs $(M, w)$ s.t. $<M, w> \leq |x| + c$, use oracle of $A_{\text{pseudocode}}$ to see if $M$ terminates on $w$ and if it does then run $M$ on $w$ and check whether $M$ leaves $x$ on tape. Length of a shortest such pair $(M, w)$ is $K(x)$. 

(b) 6.16 Proof by contradiction. Suppose that the set $A$ of incompressible strings is decidable. Then $A$ has a decider $M$, and there is an enumeration $f : A \rightarrow N$ such that $f(w_1) = 1, f(w_2) = 2, f(w_3) = 3 \ldots$ where $w_i$ is the shortest string in $A$, $w_2$ is the second shortest one, $w_3$ is the third shortest etc. Since $A$ is infinite there is a string $x \in A$ such that $|x| > |<M,x>| + \log |x| + c$. This string $x$ has a shorter description $<M', f(x)>$, where $M'$ is a machine which will output $x$ on input $f(x)$. This machine $M'$ operates as follows: run $M$ on each string $y$ in lexicographic order. Stop when $f(x)$ strings has been accepted by $M$, output current $y$. Contradiction, since $x$ is incompressible.
(c) 6.17

Proof by contradiction. Suppose that the set of incompressible strings contains an infinite recursively enumerable (Turing-recognizable) subset $A$. Then $A$ is recognized by a machine $M$ and there is an enumeration $f : \mathbb{N} \to A$ such that $f(1) = w_1, f(2) = w_2, f(3) = w_3 \ldots$ where $w_1$ is the first enumerated string in $A$, $w_2$ is the second enumerated string, $w_3$ is the third enumerated string etc. Since $A$ is infinite there is a string $x \in A$ such that $|x| > |<M>| + \log|x| + c$. This string $x$ has a shorter description $<M', f^{-1}(x)>$, where $M'$ is a machine which will output $x$ on input $f^{-1}(x)$. (This machine $M'$ operates as follows: run an enumerator until $f^{-1}(x)^{th}$ string has been produced.) Contradiction, since $x$ is incompressible.

4. We will reduce $A_{\text{pseudocode}}$ to HALT (sometimes referred to in class as the set $H$, i.e., the set corresponding to the halting problem). We construct $f : (M, x) \to M'$ where $(M, x) \in A_{\text{pseudocode}} \iff M' \in \text{HALT}$. Machine $M'$ is constructed as follows: ignore all input, simulate $M$ on $x$, if $M$ returns yes then accept. If $M$ returns no then loop indefinitely.

Note that

if $M$ does not halt on $x$ then $(M, x) \notin A_{\text{pseudocode}}$ and $M' \notin \text{HALT}$
if $M$ does halt on $x$ and rejects then $(M, x) \notin A_{\text{pseudocode}}$ and $M' \notin \text{HALT}$
if $M$ does halt on $x$ and accepts then $(M, x) \in A_{\text{pseudocode}}$ and $M' \in \text{HALT}$. 

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