A graph is uniquely localizable in $\mathbb{R}^d$ if it has a unique realization in $\mathbb{R}^d$ and no nontrivial other realizations in dimension $> d$.

**Theorem (Yinyu Ye):** There exists a polynomial algorithm to find a real-alization embedding of a $d$-uniquely localizable graph.

**Proof:** Setting up a semidefinite program

**Lemma 1:** If there is no nontrivial realization in dimension $> d$ and there exists a realization in $\mathbb{R}^d$, then the realization is unique.

**Lemma 2:**
1. Uniquely localizable in $d \iff$
2. Max-rank completion $M$ to given partial matrix $M_G$ is of rank $d \iff$
3. Solution $X$ to given SDP satisfies $M = X^T X$

**Open Q:** Give a graph theoretical characterization of $d$-unique localizability.

### 1 theorem to cover

**1.1 Schonberg’s theorem:** A metric space $\delta$ is of $2$-negative type $\iff$ it is isometrically embeddable in a Hilbert space.

Recall: For $1 \leq p \leq 2$, a metric space has $p$-negative type iff isometrically embeddable in $L_p$.

**1.2 Lennort Tonge Weston theorem**

**1.2.0.3 $p$-negative type = generalized roundness $p$.**

**1.3 Bourgain’s theorem:** Tree metrics are not isometrically embeddable in $\mathbb{R}_i$.

Recall: They are embeddable in $\mathbb{R}_{1}$. 
1.4 Weston’s theorem
1.4.0.4 Any finite metric space w/n pts is isometrically embeddable in $L_p^n$ for some $p > 0$ ($p$ depending only on $n$)

1.5 For $p > 2, n > 3$, $L_p^n$ does not have $q$ -- negative type for any $q > 0$.

2 agenda
2.1 folklore theorem-symmetric positive definite matrix
cooley-menger conditions

2.2 Laman’s theorem-partial metric space, embeddability into $\mathbb{R}^2$, finite # of embeddings

2.3 Jackson & Jordan’s theorem-unique embeddability into $\mathbb{R}^2$, generic theoretic characterization.

2.4 $d$ -- realizability (generic graph theoretic characterization)
connelly slaughter- $d = 3$. $d = 4,5$ open

2.5 realization of $d$ unique localizable graphs (Yiu-yu Ye)
polynomial time-semidefinite program

3 Theorem
3.1 Schönberg’s theorem

3.1.1 A finite distance space $\delta(\forall i,j, \delta_{ij} \geq 0, \delta_{ii} = 0, \delta_{ij} = \delta_{ji})$ is isometrically embeddable in Hilbert space iff $\forall n$ points, $n \geq 2$ & for all real values $\alpha_1 \ldots \alpha_n$ $\sum \alpha_i = 0$, $\sum \delta^{2(p)}_{ij} \alpha_i \alpha_j \leq 0 \Rightarrow$ has $2(p)$--negative type.

folklore theorem: A finite distance $\delta$ is isometrically embeddable in Hilbert space iff the metric matrix $M_{ij} = \frac{1}{2}(\delta^2_{ii} + \delta^2_{jj} - \delta^2_{ij})$ is positive semi-definite. \footnote{\forall \alpha \in \mathbb{R}^n, \alpha^T M \alpha \geq 0}

Fact: $\forall m$ & for any submatrix $M'$ of dimension $m$ (w/duplication of rows and columns allowed), $\Leftrightarrow \forall \alpha \in \mathbb{R}^n, \alpha^T M' \alpha \geq 0$. 

\footnote{\forall \alpha \in \mathbb{R}^n, \alpha^T M \alpha \geq 0}$
Theorem: a separable (countable dense set) distance space $\delta$ is embeddable into Hilbert space iff the family of functions $e^{-\lambda^2}$ is positive definite in $\delta$.

**Def:** A function $g(\text{real, continuous})$ is positive definite over $\delta$ if $\forall \alpha \in \mathbb{R}, \sum_{i,j=1} g(\delta_{ij})\alpha_i\alpha_j \geq 0$.

### 3.1.2 Roundness

**Def1:** A metric space $(X, \delta)$ has roundness $q$, $q \in g^r(X, \delta)$, whenever $\forall a_1a_2b_1b_2 \in X$, $\delta(a_1a_2)^q + \delta(b_1b_2)^q \leq \sum_{1 \leq i,j \leq 2} \delta(a_ib_j)^q$.

**Def2:** A metric space has generalized roundness $q$ when $\forall a_1 \cdots a_n b_1 \cdots b_n \in X$,

\[
\sum_{1 \leq i < j \leq n} \delta^q(a_i a_j) + \sum_{1 \leq i < j \leq n} \delta^q(b_i b_j) - \sum_{1 \leq i,j \leq n} \delta^q(a_i b_j) \leq 0 \tag{2}
\]

**Theorem I:** A metric space $(X, \delta)$ has generalized roundness $p \iff$ it has generalized roundness $q, \forall q \leq p$.

#### 3.1.2.1 Theorem II: A metric space has $q$-negative type

$(\forall \alpha_1 \cdots \alpha_n \sum \alpha_i = 0, \forall \{x_1 \cdots x_n\} \in X, \sum \delta(x_i x_j)^q \alpha_i \alpha_j \leq 0)$ \tag{1}

$\iff$ it has generalized roundness $q$.

**Lemma:** For a metric space $(X, \delta)$ the following are equivalent.

1. $q \in g^r(X, \delta)$, ie. $(X, \delta)$ has generalized roundness $q$.

2. $\forall n \in \mathbb{N}$ & all $\{x_1 \cdots x_n\} \subseteq X$, & all $w_1 \cdots w_n s_1 s_n, \sum w_i = \sum s_i (=1$ if normalization is needed),

\[
\sum \delta(x_i x_j)^q (w_i - s_i) (w_j - s_j) \leq 0 \tag{3}
\]

#### 3.1.2.1.1 Proof of Theorem II:

$\Rightarrow$ let $(X, \delta)$ be $q$-negative type.

$x_1 = a_1, x_3 = a_2, \ldots x_{2n-1} = a_n$

$x_2 = b_1, x_4 = b_2, \ldots x_{2n} = b_n$ and $\alpha_k = (-1)^k \forall 1 \leq k \leq 2n$

\[2\text{[set of pts, pairwise distance]}\]
since \( \sum_{1 \leq i,j \leq n} \delta(x_ix_j)^q \alpha_i \alpha_j \leq 0 \), sum over:

1. \( i,j \) (both odd),
2. \( i,j \) (both even),
3. \( i \) odd, \( j \) even,
4. \( i \) even, \( j \) odd

\[
0 \geq 2(2) = \sum_{1 \leq i,j \leq n} \delta^q(a_i a_j) + \delta^q(b_i b_j) - 2\delta^q(a_i b_j)
\]

therefore
\[
0 \geq \sum_{1 \leq i < j \leq n} \delta^q(a_i a_j) + \sum_{1 \leq i < j \leq n} \delta^q(b_i b_j) - \sum_{1 \leq i,j \leq n} \delta^q(a_i b_j)
\]

\( \Leftarrow \) let \((X, \delta)\) have generalized roundness \( q \),

Take \( x_1 \cdots x_n \in X \& \alpha_1 \cdots \alpha_n \in \mathbb{R} \), satisfying \( \sum \alpha_i = 0 \),

if \( \alpha_k > 0 \) then set corresponding \( w_k = |\alpha_k|/\sum_k |\alpha_k|, s_k = 0 \)

if \( \alpha_k < 0 \) then set corresponding \( w_k = 0, s_k = |\alpha_k|/\sum_k |\alpha_k| \)

simply substitute into (3) to get (1).

3.1.2.1.2 Proof of lemma

\( \Rightarrow \) let \((X, \delta)\) have generalized roundness \( p \),

Take \( x_1 \cdots x_n \in X \& w_1 \cdots w_n, s_1 \cdots s_n \geq 0, \sum w_i = \sum s_i = N^3 \)

construct a double – \( N \) simplex

\[
a_1 = a_2 = \cdots a_{w_1} = x_1
\]

\[
a_{w_1+1} = \cdots a_{w_1+w_2} = x_2
\]

\[
a_{w_1+w_2+1} = \cdots a_{w_1+w_2+w_3} = x_3
\]

\[
b_1 = b_2 = \cdots b_{s_1} = x_1
\]

\( ^3 \) without loss we can assume these are natural numbers since the rations are dense in the reals

\[4\]
\[ b_{s_1+1} = \cdots b_{s_1+s_2} = x_2 \]

From (2) it follows:

\[ \sum_{1 \leq i,j \leq n} \delta(x_i, x_j)^p \left[ \frac{w_i w_j + s_i s_j}{N^2} \right] \leq 2 \sum_{1 \leq i,j \leq n} \delta(x_i, x_j)^p \left[ \frac{w_i w_j}{N^2} \right] \]

3.1.2.2 Theorem: For any \( p > 1 \), \( \exists \) a tree metric space \( (X, \delta) \) which has generalized roundness \(< p \).

Def1: A metric space \( (X, \delta) \) has generalized roundness \( p \) if \( \forall \forall a_1 \cdots a_n b_1 \cdots b_n \in X, \sum_{1 \leq i,j \leq n} \delta(a_i a_j)^p + \delta(b_i b_j)^p \leq \sum_{1 \leq i,j \leq n} \delta(a_i b_j)^p \)

Def2: Given a tree \( T = (V,E) \), tree metric space is defined as \( (X = \{1,2,\ldots,n\}, \delta) \), \( \delta(v_1, v_2) = \text{path length between}(v_1, v_2) \)

Cor: For any given \( p > 1 \), not all tree metrics are embeddable into \( L_p \).

3.1.2.2.1 Proof:

For any \( p > 1 \), need a tree \( T = (V,E) \), with \( m \) nodes & an \( n \) & a double simplex \( a_1 \cdots a_n b_1 \cdots b_n \in V \), s.t

\[ \sum_{1 \leq i,j \leq n} \delta(a_i a_j)^p + \delta(b_i b_j)^p \geq \sum_{1 \leq i,j \leq n} \delta(a_i b_j)^p \]
Take \( \delta(a_i a_j) = 2, \delta(a_i b_j) = 1, \delta(b_i b_j) = 0 \), then \( \left( \frac{n}{2} \right)^2 \geq n^2 \cdot 1^p \Rightarrow n \geq \frac{1}{1 - \frac{1}{2^p}}, \) here \( \varepsilon = p - 1 \)

Conjecture: Any tree metric of \( n \) points is embeddable in \( L_p \) for \( p = 1 + \varepsilon(n) \)

3.1.2.3 Theorem: Every finite metric space of \( n \) points has generalized roundness \( \geq p(n) \), where \( p(n) = \log_2(1 + \frac{\psi^2}{n}) \approx \log\left(\frac{n}{n-1}\right), \) \( v = \frac{2}{(2n)^{\psi(n)}} \), where \( \psi(1) = \psi(2) = 1, \psi(k) = \psi(k-1) + \psi(k-2) + 1 \)

Fact: This bound is quite tight.

3.1.2.3.1 Proof:

Proposition (base case of induction)

If \( (X, \delta) \) is a 4-point metric space then generalized roundness of it \( \geq 1 \)
case 1:
\[ a_1 = a_2 = \ldots a_m = x, \quad b_1 = b_2 = \ldots b_q = z \]
\[ a_{m+1} = \ldots a_n = y, \quad b_{q+1} = \ldots b_n = w \]
\[ 0 < m, q < \frac{n}{2}, \quad m(n - m) \cdot d + q(n - q) \cdot c \leq mqu + (n - m)qe + m(n - q)f + (n - m)(n - q)v \]
(this is true because of triangle inequality)

\[ \text{case 2:} \]
\[ a_1 = a_2 = \ldots a_{n_1} = z \]
\[ a_{n_1+1} = \ldots a_{n_2} = x \]
\[ a_{n_1+n_2+1} = \ldots a_{n_3} = y \]
\[ b_1 = b_2 = \ldots b_n = w \]
Lemma 1: \( n \geq 2, 0 \leq p \leq \log_2\left(\frac{n}{n-1}\right) \) and \( [a_i, b_i]_{i=1}^n \subseteq (X, \delta) \) is a given double simplex ordered s.t. \( \delta(a_1b_1) \leq \delta(a_ib_j), \forall i, j \), then for \( \forall j, 2 \leq j \leq n \), \( \delta(a_1a_j)^p \leq \frac{\delta(a_1b_1)^p}{2(n-1)} + \delta(b_1a_j)^p \)

Lemma 2: consider a double simplex \( [a_i, b_i]_{i=1}^n \subseteq (X, \delta) \) arranged so that \( \delta(a_1b_1) \leq \delta(a_ib_j), \forall i, j \). If \( 0 \leq p \leq \log_2\left(\frac{n}{n-1}\right) \) and \( p \in g.r. \), then \( g.r. \leq p \) for this particular simplex then \( p \in g.r. \).

3.1.2.4 Theorem: \( \forall p > 1, \exists \) tree metrics not embeddable in \( L_p \).

\( \forall p > 1, \exists \) tree metrics which do not have \( p \) negative type, for \( (1 \leq p \leq 2), \Leftrightarrow \) not embeddable in \( L_p \).

Conjecture: \( \exists p(n) \geq 1 \), s.t all tree metrics of \( n \) points have negative type \( \geq p(n) \)

\( \forall p > 0, \exists \) metrics spaces whose generalized roundness & negative type \( < p \Leftrightarrow \exists p(n) \) depending only on \( n \) s.t all metric spaces of \( n \) points have generalized roundness & negative type \( \geq p(n) \)

Theorem 2: \( \forall p > 2, L_p^d \) (even for \( d = 3 \)) does not have negative type \( q \) \( \forall q > 0 \)

Open Question: construct the finite double simplex that shows this. i.e that \( L_p^3 \) does not have generalized roundness or negative type \( q \).

Fact: \( L_q \) does not have \( q \) negative type for \( q > 2 \). (We know \( L_q \) has \( q \) negative type between \( 1 \leq q \leq 2 \))

Theorem: \( \exists \) an isometric embedding of \( L_2 \) in \( L_p \) \( \forall 1 \leq p \leq \infty \)

Open Question: construct versions even for finite subsets of \( L_2 \).

Theorem 1 does not imply tree metrics are not embeddable in \( L_p \) for \( p > 2 \). How about embeddability in \( L_\infty \)?

Theorem: every metric space \( n \) is embeddable in \( L_\infty^n \), where \( \|x\|_\infty = \max_i |x_i| \).

Proof: set

\[
\begin{align*}
x_i & \rightarrow d[x_i, x_1], d[x_i, x_2], \ldots d[x_i, x_i], \ldots d[x_i, x_j] \ldots d[x_i, x_n] \\
x_j & \rightarrow d[x_j, x_1], d[x_j, x_2], \ldots d[x_j, x_i], \ldots d[x_j, x_j] \ldots d[x_j, x_n], \; i < j \\
d[x_i, x_j] & \rightarrow \max(d[x_i, x_k] - d[x_j, x_k], 0) = d[x_i, x_j] \text{ because triangle inequality}
\end{align*}
\]
3.2 Realizability of Graphs

3.2.1 Main Theorem: A graph is 3-realizable ⇔ it has no minor.

Def: $d$-realizability ⇔ a constraint system has an embedding in $x$-$dim$ ⇒ embeddable in $d$-$dim$.

G is $d$-realizable if $\forall \delta(E)$ [(G, E) has embedding in $m$-$dim$ ⇒ G has an embedding in $d$-$dim$]

Def [Minor]: A minor of a graph G is the graph that transformed from a subgraph of G by:

- Edge deletion
- Edge contraction

Def [k-tree]: A graph is a $k$-tree if it can be obtained through a sequence of $k$-sum of $K_{k+1}$.

Def [partial $k$-tree]: subgraph of $k$-tree.

Theorem 1: partial $d(3)$-tree is $d(3)$-realizable.

Theorem 2: Forbidden minors of partial 3-tree is
Theorem 3: If \( G \) has a minor \( \Rightarrow G \) is not 3-realizable

Conjecture: If a graph has \( e \) edges and \( e < \frac{(d+1)(d+2)}{2} \), then \( G \) is partial \( d \)-tree. Furthermore, if \( G \) has \( e = \frac{(d+1)(d+2)}{2} \), and \( G \) is not the complete graph \( K_{d+1} \), then \( a \) is still a \( d \)-tree.