# Notes On Approximating Arbitrary Metrics by Tree Metrics

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These notes outline the basic definitions and theorems we looked at in class when studying [**?**]. An intutive sketch of the proof of the main theorem is given.

## 1 Definitions

The following where the definitions of the basic elements of the work in this paper. First, the authors define how a metric space domainates another:

**Definition 1.1.** *A metric spaces $N$ over $V$, dominates a metric space $M$ over $V$, if for every $u$,$v \in V$, $d_N(u,v) \geq d_M(u,v)$.*

Next the authors introduce the notion of $\alpha$-approximation:

**Definition 1.2.** *A metric spaces $N$ over $V$, $\alpha$-approximates a metric space $M$ over $V$, if it dominates $M$ for every $u$,$v \in V$, $d_N(u,v) \leq \alpha.d_M(u,v)$.*

**Definition 1.3.** *A set of metric spaces $S$ over $V$ $\alpha$-probablistically-approximates a metric space $M$ over $V$, if every metric space in $S$ dominates $M$ and there exists a probability distribution over metric spaces $N \in S$ such that for every $u$,$v \in V$, $E(d_N(u,v)) \leq \alpha.d_M(u,v)$.*

**Definition 1.4.** *A set of metric spaces $S$, over $V$, $\alpha$-distributionally-approximates a metric space $M$ over $V$, if every metric space in $S$ dominates $M$ and for every nonnegative weight function $\mu$ on pairs of $V$ (and assume $\sum \mu(u,v) = 1$) there exists a metric space $N \in S$ such that $\sum \{u,v\} \in V^2 \mu(u,v) \frac{d_N(u,v)}{d_M(u,v)} \leq \alpha$.*

Note that the definitions of $\alpha$-distributionally-approximation and $\alpha$-probablistically-approximation are equivalent. This can be seen by viewing the problem of obtaining a probabilistic approximation of a metric space as a two person zero sum game where the "metric player" chooses a metric spaces $N$ out of a fixed set $S$ and the "edge player" chooses a pair of vertices with the payoff function of the "edge player" defined to be the distortion of the distance between the pair. The equality then follows from applying the von Neumann minimax principle of game theory.

Then the authors introduce the basic metric of the paper:

**Definition 1.5.** *A tree metric (or additive metric) over $V$ is a metric space corresponding to a wieghted tree spanning $V$.*

The embedding is into a specific kind of tree called ($k$-HST):

**Definition 1.6.** *A k-heirachically well sperated tree (k-HST) is defined as a rooted weighted tree witht the following properties:*

- *The edge weigh from any node to each of it's children is the same.*

- *The edge weights along any path from the root to a leaf are decreasing by a factor of at least k.*

In the process of embedding a graph into a ($k$-HST), the graph is first embeded into an intermediate structure called an HPM:

**Definition 1.7.** *A heirachically partition metric (HPM) on a weighted connected graph $G$ is a length function $l(e)$ defined on the edges of $G$ according to the following recursive process: let $C$ be a cut in $G$. For ever $e \in C$, $l(e) \geq diam(G)$ and $l$ forms a hierarchical partition metric over each connected component recursively*

## 2 Theorems

The basic idea is that weighted graphs are approximated by HPMs and then HPMs are approximated by k-HSTs. The first theorem is going from an HPM to a $k$-HST:

**Theorem 2.1.** *Given weighted connected graph $G$ and an HPM $l$ of $G$. There exists a $k - HST$ that dominates $G$ and for every u,v $\in V$, $d_T(u,v) \leq \frac{k^2}{k-1} l(u,v)$. Note that this HST has diameter propotional to that of G. Moreover the HST can be constructed is polynomial time.*

To get from an HPM to a k-HST, we let $l$ be an HPM of $G$. We build a $k$-HST, $T$, recursively as follows. Consider the hireachical partition metric. Every vertex in the tree correspnds to a subgraph in the construction of the partition. The root corresponds to the entire graph. And $L = \delta$. At a certain level in the recursion the current sub-graph $H$ corrspnds to a leaf on the current tree $T$. If diam$(H) \leq L/k$ then let $v$ be the leaf corresponding to the subgraph $H$ and decrease $L$ by a factor of $k$. Otherwise let $v$ be the parent of that leaf. Let $C$ be the cut that defines the HPM and consider the connected components that are left after removing the cut edges. Denote by $v$ the child for each of these clusters. The weight of the tree edge of each of the children is equal to $L/2$. This process is continued recursively on each cluster. It is then shown that the distances in the HST are dominating the distances according to the orginial metric and are bounded above by $k/(k-1)$ times the distance on the HPM.

**Theorem 2.2.** *Every weighted connected graph $G$ can be $\alpha$-probabilstically- approximated, or $\alpha$-distrbutionally- approximated, by a set of $HPMs$ where $\alpha = O(\log n \log logn)$. Moreover the probability distribution computation and the $HPM$ construction are polynomial time.*

Mapping a wieghter graph to an HPM requires specifying a partition scheme. This is done in a recursive fashion. First, the graph is slightly modfied by first contracting edges $e$ where $w(e)$ is smaller than the diameter of $H$ divided by $2 * |E(H)|$. Then this fix a vertex $v$ and partition the current graph into two subgraphs based on $d(u,v)$. The bound follows by induction. The partition is done by choosing an interger $g$ in the range between 1 and $|E(H)|$. And let the cut-off radius be $\frac{g}{|E(H)|}$ times the diameter of the current graph. $g$ is choosen based on solving a linear program with a linear equation for each possible value of $g$.,

## References

[1] Yair Bartal *On Approximating Arbitrary Metrics by Tree Metrics*