Lecture :3D Realizability

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We will present that the forbidden minors for 3-realizable graph are K_5 and the octahedron. You can directly go to Theorem 13 and if necessary refer to the definitions, lemmas and theorems before it.

Definition 1 *A graph G is **d-realizable** if, given any realization $p_1, \cdots, p_n$ of the graph in some finite dimensional Euclidean space, there exists a realization $q_1, \cdots, q_n$ in E^d with the same edge lengths: $|p_i - p_j| = |q_i - q_j|$ for all $\{i, j\} \in E(G)$.*

Note that in the definition of **d-realizable** allows edges to have length zero. Also note that **d-realizable** is a property of graphs.

Definition 2 *Let G_1 and G_2 be two graphs, both containing a K_m as a subgraph. The **m-sum of** G_1 **and** G_2, denoted $G_1 \bigoplus G_2$, is the graph obtained by identifying the two K_m's.*

Definition 3 *A graph is **m-tree** if it can be obtained through a sequence of m-sums of K_{m+1}'s. A graph is a **partial m-tree** if it is a subgraph of a m-tree.*

Theorem 4 *All partial d-trees are d-realizable.*

Definition 5 *A minor of a graph G is any graph obtained from G by a sequence of*

- *edge deletions and*
- *edge contractions (identify the two vertices belonging to an edge and then remove any loops or multiple edges)*

Theorem 6 *If a graph G is d-realizable and H is a minor of G, then H is d-realizable.*

Sketch of Proof: Zero length edges are allowed. ■

Theorem 7 *The forbidden minors for partial 3-trees are K_5, the 1-skeleton of the octahedron ($K_{2,2,2}$), V_8 and $C_5 \times C_2$(see Figure 1).*

Theorem 8 *K_5 is not 3-realizable.*

Actually, K_5 is over-constrained in 3D but under-constrained in 4D.

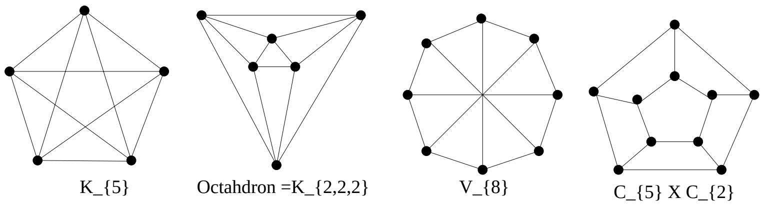

Figure 1: Forbidden minors for partial 3-trees

Theorem 9 *The 1-skeleton of the octahedron ($K_{2,2,2}$) is not 3-realizable.*

Sketch of Proof:

Assign the following values for K_5 shown in Figure 2:$d_{12} = 1$,$d_{23} = 1$,$d_{13} = 2$, $d_{14} = \sqrt{2}$,$d_{34} = \sqrt{2}$, $d_{15} = d_{25} = d_{45} = 1$,$d_{26} = d_{36} = d_{46} = 1$. In order to have a solution in 3D, d_{56} has to be $\sqrt{2}$ or 2. So if we let $\sqrt{2} < d_{56} < 2$, then it has no embedding in 3D but have infinite many solutions in 4D.

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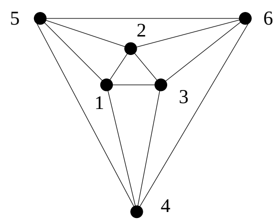

Figure 2: $K_{2,2,2}$ is not 3-realizable

The graphs V_8 and $C_5 \times C_2$ are 3-realizable. This leaves the possibility that there are other graphs which are not 3-realizable but do not have K_5 or the octahedron as a minor. The following discussion will eliminate this possibility by showing that any graph containing V_8 or $C_5 \times C_2$ as a minor either contains K_5 or octahedron as a minor or is 3-realizable.

Lemma 10 *If any edge is added between non-adjacent vertices of V_8, the resulting graph has K_5 as a minor.*

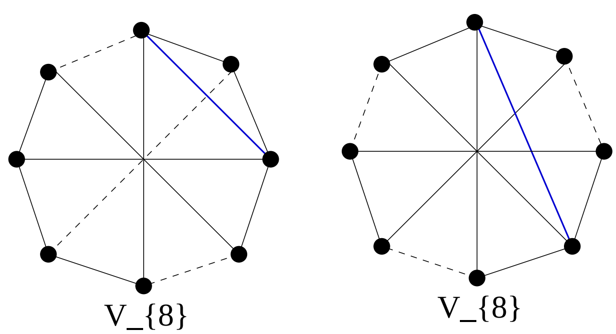

Figure 3: Graphs of V_8 with an added edge contract to K_5

Sketch of Proof: There are two ways to add an edge to V_8 up to graph isomorphism. Refer to Figure 3, if we contract the dotted edges, the resulting graph is K_5.

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Lemma 11 *If any edge is added between non-adjacent vertices of $C_5 \times C_2$, the resulting graph has either the octahedron or K_5 as a minor.*

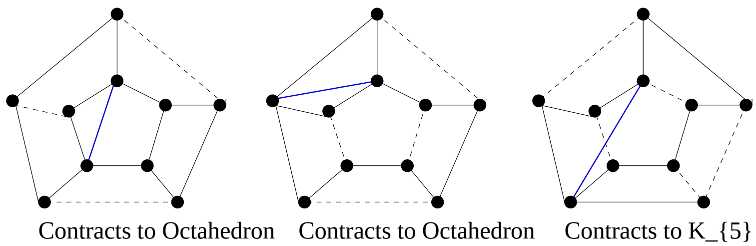

Figure 4: Graphs of $C_5 \times C_2$ with an added edge contract to octahedron or K_5

Sketch of Proof:

There are three ways to add an edge to $C_5 \times C_2$ up to graph isomorphism. Refer to Figure 4, if we contract the dotted edges, the resulting graph is either octahedron or K_5.

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A graph H is a **subdivision** of a graph G if H can be obtained from G by replacing every edge $\{i, j\}$ of G with a path from vertex i to vertex j.

Lemma 12 *Let H be a graph whose vertices are of maximum degree 3. If a graph G has H as a minor, then G contains a subdivision of H as a subgraph.*

Theorem 13 *The forbidden minors for 3-realizable graph are K_5 and the octahedron.*

Sketch of Proof:

By Theorem 4, all partial 3-trees are 3-realizable.That is, by Theorem 7, if a graph has no minor of K_5, $K_{2,2,2}$,V_8 or $C_5 \times C_2$, it is 3-realizable.

According to Theorem 6 together with Theorem 8 and Theorem 9, if a graph has a minor K_5 or $K_{2,2,2}$ then this graph is not 3-realizable.

Now to finish the proof, let's consider the graphs which have minor V_8 or $C_5 \times C_2$ but do not have minor K_5 or $K_{2,2,2}$. We will prove in this case the graphs are 3-realizable by showing it is a subgraph of 2-sum or 3-sum of 3-realizable graphs.

Firstly consider the case that G has V_8 as a minor. Note that any vertex in V_8 has degree right 3, so by Lemma 12, G has a subgraph graph which is a subdivision of V_8 and we denote it by H. Remove H from G, we will prove that each connected component in $G \setminus H$ is connected in G to exactly one of the subdivided edge of H. One connected component may only connect to one vertex which is the end vertex of 2 or 3 subdivided edges and we assign any subdivided edge to this component.

If a component did connect to two subdivided edges, say $\{i, j\}$ and $\{k, l\}$. Since V_8 contains no triangles, two of the four relevant vertices(say i and k) must be non-adjacent in V_8. The subdivided edges can be contracted in G so that the path goes from i to k, which contradicts Lemma 10.

Meanwhile we have the same argument with the case that G has $C_5 \times C_2$ as a minor.

Thus, we can assign a subdivided edge $\{i, j\}$ to each of the components.

For any subdivided edge $\{i, j\}$,we can add an edge $\{i, j\}$ to G if edge $\{i, j\}$ is not in G. So we can get a new graph by adding this kind of edges. This new graph is the 2-sum along the edge $\{i, j\}$ of smaller graphes such as V_8,$C_5 \times C_2$ or partial 3-tree. So this new graph is 3-realizable. G is a subgraph of this new graph, so G is 3-realizable. Now we discussed all the cases and finished the proof.

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A characterization of 3-realizable graphs is: every 3-realizable graph is a subgraph of a graph that can be obtained by a sequence of 3-sums and 2-sums involving K_4,V_8 and $C_5 \times C_2$.

References

[1] Connelly, R., and Sloughter, M. "Realizability of Graphs," *http://www.math.cornell.edu/~connelly/realizability.new.pdf.*