Geometric Constraint II (Oct 3-19)

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Oct 23, 2006

0.0.0.1 A graph is uniquely localizable in $\mathbb{R}^d$ if it has a unique realization in $\mathbb{R}^d$& no nontrivial other realizations in dimension $> d$.

0.0.0.2 Theorem(Yinyu Ye): $\exists$a polynomial algorithm to find a realization embedding of a d-uniquely localizable graph.

Proof: setting up a semidefinite program

Lemma1: if $\nexists$a nontrivial realization in dimension $> d$ & $\exists$a realization in $\mathbb{R}^d$, then the realization is unique.

Lemma2:

1. uniquely localizable in $d \Longleftrightarrow$
2. max-rank completion M to given partial matrix M_G is of rank $d \Longleftrightarrow$
3. solution X to given SDP satisfies $M = X^T X$

OpenQ: Give a graph thearetic characterization of d-unique localizability.

1 theorem to cover

1.1 Schonberg's theorem: A metric space δ is of 2-negative type $\Leftrightarrow$it is isometrically embeddable in a Hilbert space.

recall: for $1 \leq p \leq 2$, a metric space has $p-negative$ type iff isometrically embeddable in L_p.

1.2 Lennort Tonge Weston theorem

1.2.0.3 $p-negative$ type = generalized roundness p.

1.3 Bourgain's theorem: Tree metrics are not isometrically embeddable in $\mathbb{R}_i$.

recall: they are embeddable in $\mathbb{R}_1$.

1.4 Weston' s theorem

1.4.0.4 Any finite metric space w/n pts is isometrically embeddable in $L_p$for some $p > 0$ (p depending only on n)

1.5 For $p > 2$,$n > 3$, L_p^n does not have $q - negative$ type for any $q > 0$.

2 agenda

2.1 folklore theorem-symmetric positive definite metric matrix

coyley-menger conditions

2.2 Laman's theorem-partial metric space, embeddability into $\mathbb{R}^2$,finite #of embeddings

2.3 Jackson & Jordan's theorem-unique embeddability into $\mathbb{R}^2$, generic theoretic characterization.

2.4 $d - realizability$ (generic graph theoretic characterization)

connely sloughter- $d = 3$. $d = 4, 5$ open

2.5 realization of d unique localizable graphs (Yiu-yu Ye)

polynomial time-semidefinite program

3 Theorem

3.1 Schonberg's theorem

3.1.1 A finite distance space $\delta(\forall i, j,\delta_{ij}\geq 0,\delta_{ii}= 0,\delta_{ij} = \delta_{ji})$ is isometrically embeddable in Hilbert space iff $\forall_{n\,points}, n \geq 2$& for all real values $\alpha_1 \dots \alpha_n$ $\sum \alpha_i = 0$, $\sum \delta_{ij}^{2(p)} \alpha_i \alpha_j \leq 0$ $\Rightarrow\delta$has $2(p) - negative$ type.

folklore theorem: A finite distance δ is isometrically embeddable in Hilbert space iff the metric matrix $M_{ij} = \frac{1}{2}(\delta_{0i}^2 + \delta_{0j}^2 - \delta_{ij}^2)$ is positive semedefinite.[1]

Fact:$\forall m$ & for any submatrix $M'$of dimension m(w/duplication of rows and columns allowed), $\Leftrightarrow \forall \alpha \in \mathbb{R}^n, \alpha^T M' \alpha \geq 0$.

[1] $\Leftrightarrow \forall \alpha \in \mathbb{R}^n, \alpha^T M \alpha \geq 0$

Theorem: a separable(countable dense set) distance space $\delta$is embeddable into Hilbert space iff the family of functions $e^{-\lambda t^2}$ is positive definite in δ.

Def: A function g(real, continuous) is positive definite over δ if $\forall \alpha \in \mathbb{R}, \sum\limits_{i,j=1} g(\delta_{ij})\alpha_i\alpha_j \geq 0$.

3.1.2 Roundness

Def1: A metric space $(X, \delta)^2$ has roundness q, $q \in r(X, \delta)$, whenever $\forall a_1a_2b_1b_2 \in X,\ \delta(a_1a_2)^q + \delta(b_1b_2)^q \leq \sum\limits_{1\leq i,j\leq 2} \delta(a_ib_j)^q$

Def2: A metric space has generalized roundness q when $\forall a_1 \cdots a_n b_1 \cdots b_n \in X$,

$$\sum_{1\leq i<j\leq n} \delta^q(a_ia_j) + \sum_{1\leq i<j\leq n} \delta^q(b_ib_j) - \sum_{1\leq i,j\leq n} \delta^q(a_ib_j) \leq 0 \tag{2}$$

Theorem I: A metric space (X, δ) has generalized roundness p $\Leftrightarrow$it has generalized roundness q,$\forall q \leq p$

3.1.2.1 Theorem II: A metric space has $q-negative$ type

$$(\forall \alpha_1 \ldots \alpha_n\ \ \sum \alpha_i = 0, \forall \{x_1 \cdots x_n\} \in X,\ \ \sum \delta(x_ix_j)^q \alpha_i\alpha_j \leq 0\ ) \tag{1}$$

$\Leftrightarrow$ it has generalized roundness q.

Lemma: For a metric space(X, δ) the following are equivalent.

1. $q \in g^r(X, \delta)$, ie. (X, δ)has generalized roundness q.
2. $\forall n \in \mathbb{N}$& all $\{x_1 \cdots x_n\} \subseteq X$, & all $w_1 \cdots w_n s_1 s_n$,$\sum w_i = \sum s_i(= 1$ if normalization is needed),

$$\sum \delta(x_ix_j)^q \underset{1\leq i,j\leq n}{(w_i - s_i)}(w_j - s_j) \leq 0 \tag{3}$$

3.1.2.1.1 Proof of Theorem II:

$\Rightarrow$**let** (X, δ) **be** $q - negative$ **type.**

$$x_1 = a_1, x_3 = a_2, \cdots x_{2n-1} = a_n$$

$x_2 = b_1, x_4 = b_2, \cdots x_{2n} = b_n$ **and** $\alpha_k = (-1)^k\ \forall 1 \leq k \leq 2n$

[2](set of pts,pairwise distance)

since $\sum\limits_{1\le i,j\le n}\delta(x_ix_j)^q\alpha_i\alpha_j\le 0$**, sum over :**

1. i,j (both odd),
2. i,j (both even),
3. i odd, j even,
4. i even, j odd

$0\ge 2(2)=\sum\limits_{1\le i,j\le n}\delta^q(a_ia_j)+\delta^q(b_ib_j)-2\delta^q(a_ib_j)$

therefore $0\ge\sum\limits_{1\le i<j\le n}\delta^q(a_ia_j)+\sum\limits_{1\le i<j\le n}\delta^q(b_ib_j)-\sum\limits_{1\le i,j\le n}\delta^q(a_ib_j)$

$\Leftarrow$**let** (X,δ) **have generalized roundness** q**,**

Take $x_1\cdots x_n\in X$ **&** $\alpha_1\cdots\alpha_n\in\mathbb{R}$**, satisfying** $\sum\alpha_i=0$**,**

if $\alpha_k>0$ **then set corresponding** $w_k=|\alpha_k|/\sum\limits_k|\alpha_k|,s_k=0$

if $\alpha_k<0$ **then set corresponding** $w_k=0,s_k=|\alpha_k|/\sum\limits_k|\alpha_k|$

simply substitute into (3) **to get** (1)**.**

3.1.2.1.2 Proof of lemma

$\Rightarrow$**let** (X,δ) **have generalized roundness** p**,**

Take $x_1\cdots x_n\in X$ **&** $w_1\cdots w_n,s_1\cdots s_n\ge 0$**,** $\sum w_i=\sum s_i=N^3$

construct a $double-N$ **simplex**

$a_1=a_2=\cdots a_{w_1}=x_1$

$a_{w_1+1}=\cdots a_{w_1+w_2}=x_2$

$a_{w_1+w_2+1}=\cdots a_{w_1+w_2+w_3}=x_3$

$b_1=b_2=\cdots b_{s_1}=x_1$

[3]without loss we can assume these are natural numbers since the rations are dense in the reals

$b_{s_1+1} = \cdots b_{s_1+s_2} = x_2$

from (2) **it follows:**

$$\sum_{1\le i,j\le n} \delta(x_i,x_j)^q \left[\frac{w_iw_j+s_is_j}{N^2}\right] \le 2\sum_{1\le i,j\le n} \delta(x_i,x_j)^q \left[\frac{w_is_j}{N^2}\right]$$

3.1.2.2 Theorem: For any $p>1$, $\exists$ **a tree metric space** (X,δ) **which has generalized roundness** $< p$.

Def1: A metric space (X,δ) **has generalized roundness** p **if** $\forall n \forall a_1\cdots a_n b_1\cdots b_n \in X$, $\sum_{1\le i<j\le n} \delta(a_ia_j)^p + \delta(b_ib_j)^p \le \sum_{1\le i,j\le n} \delta(a_ib_j)^p$

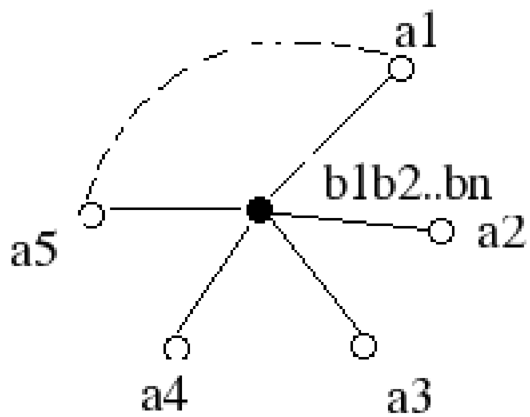
Def2: Given a tree $T=(V,E)$,**tree metric space is defined as** $(X=V,\delta)$, $\delta(v_1,v_2) = path\ length\ between(v_1,v_2)$

Cor: For any given $p>1$, **not all tree metrics are embeddable into** L_p.

3.1.2.2.1 Proof:

For any $p>1$, **need a tree** $T=(V,E)$, **with** m_p**nodes & an** n **& a double simplex** $a_1\cdots a_n b_1\cdots b_n \in V$, **s.t**

$$\sum_{1\le i<j\le n} \delta(a_ia_j)^p + \delta(b_ib_j)^p \ge \sum_{1\le i,j\le n} \delta(a_ib_j)^p$$



Take , $\delta(a_i a_j)=2, \delta(a_i b_j)=1, \delta(b_i b_j)=0$, **then** $\binom{n}{2} 2^p \geq n^2 \cdot 1^p \Rightarrow n \geq \frac{1}{1-2^{-\varepsilon}}, here\ \varepsilon = p-1$

Conjecture: Any tree metric of n points is embeddable in $L_p$for $p = 1+\varepsilon(n)$

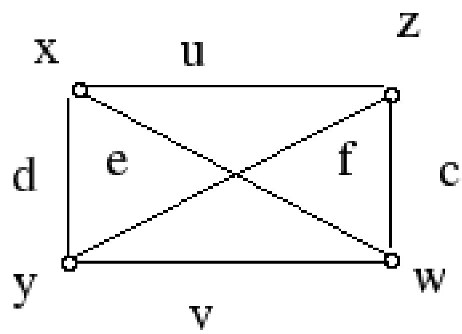
3.1.2.3 Theorem: Every finite metric space of n points has generalized roundness $\geq p(n)$, where $p(n) = log_2(1+\frac{\upsilon^2}{4}) \approx log(\frac{n}{n-1}),\ \upsilon = \frac{2}{(2n)^{\psi(n)}}$, where $\psi(1)=\psi(2)=1, \psi(k)=\psi(k-1)+\psi(k-2)+1$

Fact: This bound is quite tight.

3.1.2.3.1 Proof:

Proposition (base case of induction)

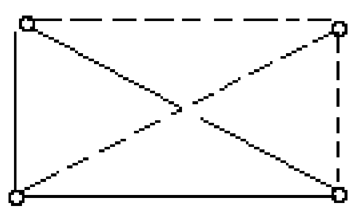
If $(X,\delta)$is a $4-point$ metric space then generalized roundness of it ≥ 1



case 1: , $a_1 = a_2 = \cdots a_m = x$, $b_1 = b_2 = \cdots b_q = z$

$a_{m+1} = \cdots a_n = y$,$b_{q+1} = \cdots b_n = w$

$0 < m, q < \frac{n}{2}$, $m(n-m) \cdot d + q(n-q) \cdot c \leq mqu + (n-m)qe + m(n-q)f + (n-m)(n-q)v$ (this is true because of triangle inequality)



case2:

$a_1 = a_2 = \cdots a_{n_1} = z$

$a_{n_1+1} = \cdots a_{n_2} = x$

$a_{n_1+n_2+1} = \cdots a_{n_3} = y$

$b_1 = b_2 = \cdots b_n = w$

Lemma1: $n \geq 2$, $0 \leq p \leq log_2(\frac{n}{n-1})$ **and** $[a_i, b_i]_{i=1}^n \subseteq (X, \delta)$ **is a given double simplex ordered s.t** $\delta(a_1 b_1) \leq \delta(a_i b_j), \forall i, j$ **,then for** $\forall j,\ 2 \leq j \leq n\ \ \delta(a_1, a_j)^p \leq \frac{\delta(a_1 b_1)^p}{2(n-1)} + \delta(b_1 a_j)^p$

Lemma2: consider a double simplex $[a_i, b_i]_{i=1}^n \subseteq (X, \delta)$ **arranged so that** $\delta(a_1 b_1) \leq \delta(a_i b_j), \forall i, j$**. if**$0 \leq p \leq log_2(\frac{n}{n-1})$ **and** $p \in g.r\,[a_i, b_i]_{i=2}^n\,(internal\ edge)^p \leq (external\ edge)^p$**for this particular simplex then** $p \in g.r\,[a_i, b_i]_{i=1}^n$

Theorem: If $[a_i, b_i]_{i=1}^n \subseteq (X, \delta)$ **is a given** $double - n - simplex$**, then** $p \in g.r\,[a_i, b_i]_{i=1}^n\,\forall p$**,** $w.t.\ 0 \leq p \leq log_2(1 + \frac{1}{2(n-1)})$

3.1.2.4 Theorem: $\forall p > 1$, $\exists$tree metrics not embeddable in L_p.

$\forall p > 1, \exists$ **tree metrics which do not have** $p - negative$ **type,** $for\ (1 \leq p \leq 2)$,$\Leftrightarrow$**not embeddable in** L_p**.**

Conjecture: $\exists p(n) \geq 1$, s.t all tree metrics of n points have negative type $\geq p(n)$

$\forall p > 0$, $\exists$**metrics spaces whose generalized roundness & negative type** $< p \Leftrightarrow \exists p(n)$ **depending only on** n **s.t all metric spaces of** n **points have generalized roundness & negative type** $\geq p(n)$

Theorem2: $\forall p > 2$, L_p^d(even for $d = 3$) does not have negative type $q\ \forall q > 0$

OpenQuestion: construct the finite double simplex that shows this. i.e that L_p^3 does not have generalized roundness or negative type q.

Fact: L_q does not have q negative type for $q > 2$. (We know L_q has $q-negative$ type between $1 \leq q \leq 2$)

Theorem: $\exists$ an isometric embedding of L_2 in L_p $\forall 1 \leq p \leq \infty$

OpenQuestion: construct versions even for finite subsets of L_2.

Theorem 1 does not imply tree metrics are not embeddable in L_p **for** $p > 2$**. How about embeddability in** L_∞**?**

Theorem: every metric space n is embeddable in L_∞^n, where $\|x\|_\infty = \max_i |x_i|$.

Proof: set

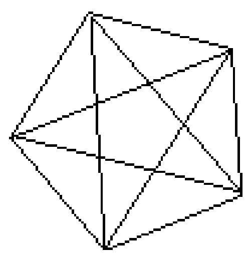
$$x_i \to d\,[x_i x_1]\,, d\,[x_i x_2] \cdots d\,[x_i x_i] \cdots d\,[x_i x_j] \cdots d\,[x_i x_n]$$

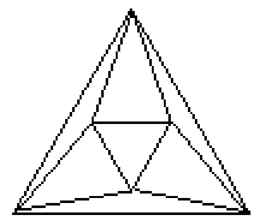
$$x_j \to d\,[x_j x_1]\,, d\,[x_j x_2] \cdots d\,[x_j x_i] \cdots d\,[x_j x_j] \cdots d\,[x_j x_n],\ i < j$$

$$d\,[x_i x_j] \to max(d\,[x_i x_k] - d\,[x_j x_k]\,, d\,[x_i x_j]) = d\,[x_i x_j]\ \ because\ triangle\ inequality$$

3.2 Realizability of Graphs

3.2.1 Main Theorem: A graph is $3-realizable$ $\Leftrightarrow$it has no minor





Def: $d-realizability$$\Leftrightarrow$a constraint system has an embedding in $x-dim$$\Rightarrow$embeddable in $d-dim$.

G is $d-realizable$ if $\forall\delta(E)$ $[(G,E)$ has embedding in $m-dim$$\Rightarrow$$G$ has an embedding in $d-dim]$

Def [Minor]: A minor of a graph G is the graph that transformed from a subgraph of G by:

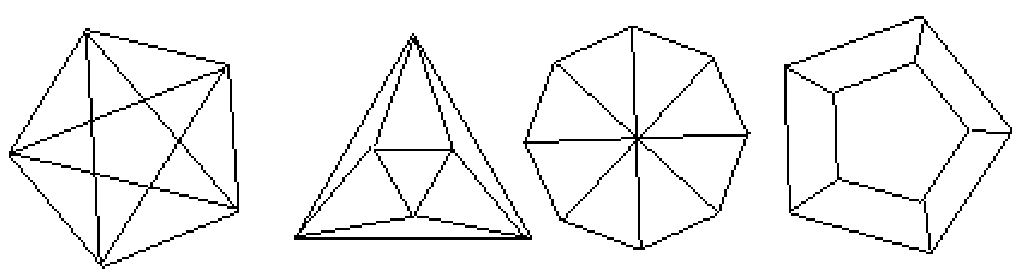
- Edge deletion
- Edge contraction

Def $[k-tree]$: A graph is a $k-tree$ if it can be obtained through a sequence of $k-sum$ of $K'_{k+1}s$.

Def [partial $k-tree$]: subgraph of $k-tree$.

Theorem1: partial $d(3)-tree$ is $d(3)-realizable$.

Theorem2: Forbidden minors of partial $3-tree$ is



Theorem3: If G has a minor $\Rightarrow G$ is not $3 - realizable$

Conjecture: If a graph has e edges and $e < \frac{(d+1)(d+2)}{2}$, then G is partial $d - tree$. Furthermore, if G has $e = \frac{(d+1)(d+2)}{2}$, and G is not the complete graph K_{d+1}, then a is still a $d - tree$.