Geometric Constraints II

Realizability, Rigidity and Related theorems.
Embeddability of Metric Spaces

Section 1: Realizability

Realizability refers to finding an embedding. Embedding as in finding a set of Cartesian co-ordinates for the set of n given points, which have the same distances between them, as between the original . This is a problem because not all of the distances are independent, Some of them are implied from the other distances. And if this implication is contradicted by certain specified distances, then the system becomes unrealizable. In this section we investigate the realizability of systems (set of points and an associated set of distances between them) It is easy to see that if 3 points have 3 distances between them that do not satisfy the triangle inequality, then they can never be realized. This condition forms the starting point in the following. We proceed to define and prove the complete set of requirements for a system to be realizable.

Given the matrix D $d_{i,j}$ $1 \leq$ i,j $\leq$ n corresponding to a metric space, give conditions under which this matrix can be realized as pairwise distances between points in R^d

Theorem: M is positive semidefinite of rank d $\Leftrightarrow$ D can be realized in R^d

Proof : ($\Leftarrow$) Consider n+1 points in R^d with one of them being the origin. Let X be the matrix of their co-ordinates. Take the gram matrix $G = G_{i,j} =< p_i, p_j >= X^T X$ (n x n matrix). $G_{i,j} = (d^2_{o,i} + d^2_{o,i} - d^2_{i,j}$). G is positive semi-definite and has rank d;

($\Rightarrow$)Since M is positive semidefinite of rank d, there exists an orthonormal Y. $L = Y^T MY => X = L^{1/2}Y$; X has only d non zero rows. Now take the gram matrix of X, $X^T X = Y^T {L^{1/2}}^T L^{1/2} Y = Y^T Y M Y^T Y = M$. It results that $M = 1/2(|p_0 p_i|^2 + |p_0 p_j| + |p_{i,j}|^2)$ for some set of points $p_1, p_2 ...., p_n$ which form the rows of X. The realization X can be obtained in $O(n^3)$ steps.

*****Volume of n points can be obtained as a determinant. In 2D a 4 point volume has to be 0.

Necessary Conditions: *(Cayley Menger Conditions)*

Suppose you are given a distance matrix (n+1)X(n+1) in R^k space.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & ... \\ 1 & 0 & \mathrm{d}_{12} & d_{13} & ... & \mathrm{d}_{1n} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 1 & \mathrm{d}_{n1} & d_{n2} & ... & ... & ...0 \end{bmatrix}$$

Then the cayley-manger condition requires that the volume of the k+2 simplex to be zero. $\forall$ k+2 simplex spanned by $(P_1...P_{k+2})Vol_{k+2}(P_1....P_k+2 =$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & ... \\ 1 & 0 & \mathrm{d}_{12} & d_{13} & ... & \mathrm{d}_{1k+2} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 1 & \mathrm{d}_{k+21} & \mathrm{d}_{k+22} & ... & ... & ...0 \end{bmatrix} = 0$$

The other condition is that the volume of all smaller simplices formed by K+1 points in the set should be positive.

$\forall\ j < k+1$ simplex spanned by $(P_1...P_j)$ $\mathrm{Vol}_j Simplex(\mathrm{P}_1....P_j) \geq 0$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & ... \\ 1 & 0 & \mathrm{d}_{12} & \mathrm{d}_{13} & ... & \mathrm{d}_{1k+2} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 1 & \mathrm{d}_{k+21} & \mathrm{d}_{k+22} & ... & ... & ...0 \end{bmatrix} \geq 0$$

Home Work : Are the above conditions sufficient ?

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Section 2: Rigidity

The Problem of rigidity : In the following section we define and demonstrate the concept of rigidity. Given a set of pairwise distances which form a subset of the entire set of $\binom{n}{2}$ distances, We say a realizable system is rigid, if it has only countably many realization. (Countable here refers a measure zero set, another way to think of it is, a set which has a bijection from the set of Naturals). Solving for the solution using the constraints imposed by the edges, and determining if a system is rigid is not computationally feasible. So we explore alternate characterizations that will help us with establishing this. Lawman's theorem stated in this section, provides a complete solution for the 2D case. The 3D case is still an open problem, although we know certain limited results

(Eg. Any convex polyhedron with non-triangular faces is not rigid, Proof Hint : Lawman's count fails.). Rigidity has many applications like for example, determination of protein conformation from NMR data.

Some Questions:

1.Why do we think about realizations as distances (not as points)? *Hint-Theorem:* Pairwise distance polynomials form the entire generator set for RE_d.

2.What is the relation between the set of embeddings of n points in R^d and the set of pairwise distance matrices?

3.Is there a function mapping from one of these sets to the other, is it one to one ?

Klein's Program: Find polynomials that are invariant under the action of the euclidean group. (They are called the Ring of invariants of the euclidean group).

Ex: Prove that the distance between any two points in 2d is invariant under any Euclidean Transformation.

Hw: Prove the above in any dimensions.

Question: Fact: We know that nd - (d+1)/2 entries of the matrix D are needed to derive the remaining ones. Which are these distances ?

Proof for the fact: The possible movements of the vertices in d space, for a set of n vertices is nd (independent motions). However a d dimensional rigid body in d space has d translations and d(d-1)/2 rotations. The total number of allowed motions is the number of total degrees of freedom nd minus the number of rigid body motions (d + d(d-1)/2).

For 2 dimensions, we have an answer to this question.(Lamans Theorem, which says that every subgraph should have at most 2n - 3 edges for n vertices, and the entire graph should have 2n -3 edges exactly.)

Formal Statement Of Lamans Theorem: Let a graph G have exactly 2n-3 graph edges, where n is the number of graph vertices in G. Then G is "generically" rigid in R^2 *iff* $e^{'} <= 2n^{'} - 3$ *for every subgraph of G having* $n^{'}$ *graph vertices and* $e^{'}$ *graph edges.* For a proof of this refer to Geometric Constraints I Notes.

Note: In any dimension, if a body is rigid it obeys the laman count. But if it obeys the count, its not sure if it is rigid.

Question: Given such a well constrained graph, and these distances, show how to construct the remaining distances.

Question: How to construct the embedding, i.e, the co-ordinates of the other points ? Given all D: Fix 2 points and then use circles to construct the remaining.

Lemma: A transformation M satisfies d(x) = d(Mx) iff M has the property of orthonormality. ($M^T M = 1$)

Theorem: A polynomial P(x) is invariant wrt $\text{E}_d iff P$ ∈ R(d)(Pairwise dist polynomials). A configuration or an n point set in R^d is a point $\in R^n d_d$ or a distance matrix d with rank d. So the dimension of the configuration space is $nd - (d+1)C_2$;

Problem: From the distance matrix generate one point configuration. *Algorithms:* 1.Gram Orthonormalization matrices. 2.Ruler and compass construction.

Section 3: Redundant Rigidity

Redundant Rigidity: In the above section, we mentioned Rigidity as having only countably many realizations. Redundant rigidity just refines this concept, in that it talks about those systems that have just one realization. This is partially characterized by Jackson and Jordan's theorem.

Jackson and Jordans Theorem: If G = (V,E) is 1.Redundantly rigid and 2.is 3-connected then G has a unique embedding. The converse is not true for all dimensions. (For a proof of this refer to Geometric Constraints Part 1 Notes.

Definition: Redundant Rigidity: Removal of any edge maintains rigidity.
Application: Given a bag of $\binom{N}{2}$ distances, their is always one distance matrix that satisfies these distances.

Algorithm: Build an arbitrary matrix with the initial values. verify CM conditions, when false $->$ permute (rows,columns)

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Section 4: Matric Embeddings

In the following sections we introduce the concept of metric embeddings, which forms the main topic for this course. Metric embeddings are very useful in the formulation of approximation algorithms for NP complete problems. Many algorithms, for example, the TSP, become solvable in polynomial time when we are allowed the assumptions that the sites are situated in metric space. This is possible because of the metric assumption on the system, namely the triangle in-equality. It provides a transitivity structure to the problem, which makes a lot of algorithms possible. But note that this facility may come at a price of distortion, meaning, the pairwise distances of the original specifications is not strictly met in the embedding, but is approximated within a given range. In the following section we explore isometric embeddings, (without distortion), and when they are possible.

The following definition might be useful.

A metric space is a 2-tuple (X,d) where X is a set and d is a metric on X, that is, a function

d : X X to R such that d(x, y) >= 0 (non-negativity) d(x, y) = 0 if and only if x = y (identity) d(x, y) = d(y, x) (symmetry) d(x, z) <= d(x, y) + d(y, z) (triangle inequality).

The function d is also called distance function.

Schoenberg1: A metric space is embeddable into an inner product space iff it has 2-negative type.

Definition: p-negative type : A metric space $d = d_{i,j} 1 \leq i,j \leq n$ has p-negative type if for every $n \in N \forall \alpha_1 \alpha_2 \alpha_3 .... \alpha_n \in R, \sum \alpha_i = 0$ then

$\sum d_{i,j}^p \alpha_i \alpha_j \leq 0.$

Fact : A space has p-negative type $\Rightarrow$ it has q-negative type for all q $\leq$ p. Proof: Suppose for some $q < p$, the space does not have a q-negative type, but

has a p negative type. $\Rightarrow \sum d_{i,j}^q \alpha_i \alpha_j \geq 0$ for some set of alphas.But this would imply that for that same set of α_i's we have $\sum d_{i,j}^p \alpha_i \alpha_j \geq 0$ which is against our assumption. We have a contradiction.

Schoenberg2: A normed space is isometrically embedded into an inner-product space iff the metric induced by the norm has 2-negative type.

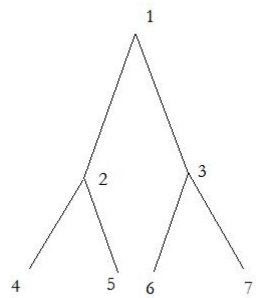
Definition: A Tree metric space is one obtained by taking vertices of a tree as points and the path length along tree edges as the metric distance between 2 points.

Fact : Any tree metric space is isometrically embeddable into L_1

BKW: A metric space is embeddable into L_p, $1 \leq p \leq 2$, iff d has p-negative type.

Fact : Tree metric space have 1-negative type.

Hint:We can prove this by assigning every point in the tree with a co-ordinate. The points numbered in this tree are assigned a co-ordinate like this



1.[000000 .... 00]
2.[100000 .....00]
3.[010000 .....00]
4.[101000 .....00]
5.[100100 .....00]
6.[101010 .....00]
7.[101001..... 00]

Notice that this kind of assignment is consistant with the tree metric, and we can always cook up co-orinates like that. And hence its always possible to embed the tree metric into L_1.

Bougain: Tree metric spaces do not have 2-negative types, hence they cannot be embedded into L_2 without distortion.

Conjecture: Tree metrics have p-negative type for $p = 1 + \epsilon$ where $\epsilon \to 0$ as $n \to \infty$

Fact: If a metric space of n points is embeddable into ${L_p}^k$, then it is embeddable into ${L_p}^{n-1}$.

Proof: Suppose you could embed n points in L_p, think of one of the points as the points as the origin and the rest of the points can form vectors with respect to this origin. Now, we can think of these n-1 vectors as forming a basis for the n-1 space (this is the max space that these vectors can span).