Geometric Constraints II
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Result 1:$\forall\ p \geq 1\ l_2$ is embeddable into l_p with constant distortion. Infact for $p = 2k$ where $k \in \mathcal{N}$, l_2 is isometrically embeddable.

A simple example of an l_2^1 embedding in l_1^2 is shown in Figure 1. Any line parrallel to the 2 co-ordinate axis is an example of an isometric embedding of l_2^1 into l_1^2. The remaining lines are examples of embeddings of l_2^1 into l_1^2 with constant distortion.

A formal statement of Result 1 is the following theorem by Dvoretzky,

Theorem 1. *For every $k \geq 2$ and every $\epsilon > 0$, there exists an $N = N(k, \epsilon)$ such that every normed space (X, p) with dim $X \geq N$ contains a k-dimensional subspace that is $\epsilon-$Euclidean.*

We now look at Figiel's proof(T. Figiel, A short proof of Dvoretzky's theorem on almost spherical sections of convex bodies) for this theorem.The proof of the above theorem follows from the results of three propositions. Before we state and prove these propositions we explain certain terms in the above theorem.

1. $N = N(k, \epsilon)$: This means that N the lower bound on the dimension of (X, p) is a function of k and ϵ.

2. $\epsilon-$Euclidean: A normed space (X, p) is $\epsilon-$Euclidean if there exists an inner-product norm, say $|.|$ and a constant C such that

$$C(1-\epsilon)|x| \leq p(x) \leq C|x|, \forall x \in X$$

We now introduce certain sets and give their geometric interpretation. Consider a normed space (X, p), such that $2 \leq dimX \leq \infty$ and the Euclidean norm $|.|$.

1. $S_X = \{x \in X : |x| = 1\}$. This is the $l_2 = 1$ ball as shown in Figure 2 for dim X =2.

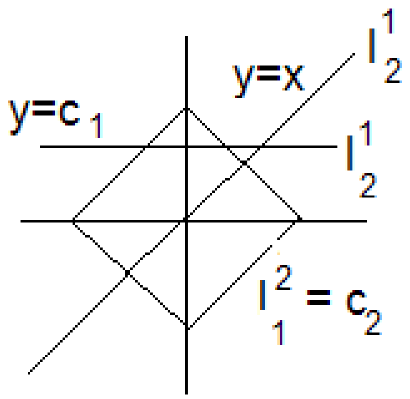

Figure 1: Examples of l_2^1 embedded in l_1^2. The $y = c_1$ line is an isometric embedding of l_2 into l_1, while the $y = x$ line is an embedding of l_2 into l_1 with a constant distortion of $\sqrt{2}$

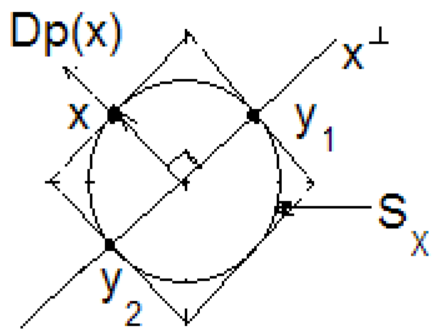

Figure 2: The figure shows the geometric interpretation of the various entities described for the 2-d scenario where p=1.

2. $x^{\perp} = \{y \in X : |x+y| = |x-y|\}, \quad \forall x \in X$. This set represents a line in 2-d(Figure 2), a plane in 3-d and a hyperplane in higher dimensions. The vector x is perpendicular to this line or plane or hyperplane(hence the set is denoted by $x^{\perp}$).

3. $\sum_X = \{(x, y) \in S_X \times S_X : y \in x^{\perp}\}$. This set consists of tuples (x, y) where x lies on the ball S_X and for every such x we pair it up with all $y's$ that lie at the intersection of the ball S_X and the hyperplane $x^{\perp}$. Figure 2 shows this for the 2-d case.

4. $\lambda_X =$ the normalized $|.|-$ rotation invariant Borel measure on S_X.

5. $\sigma_X =$ the normalized $|.|-$ rotation invariant Borel measure on $\sum_X$.

Definition of distance v(): $v(X, p, |.|) = \int_{\sum_X} (Dp(x)y)^2 p(x)^{-2} d\sigma_X(x, y)$ where $Dp(x)$ is the gradient of the p-norm at the point x. $Dp(x)y$ is the inner-product of the vectors $Dp(x)$ and y. $p(x)$ is a convex function differentiable almost everywhere. Figure 2 shows the direction of $Dp(x)$ when $p = 1$ and dimension 2.

Lets try to see if the distance $v()$ has properties of a metric,

1. Non-negativity: $v()$ is definitely non-negative since its an integral over each term squared.

2. Identity: If $p = |.|$ the integral is 0, since the direction of $Dp(x)$ will be along the vector x and x is perpendicular to the corresponding $y \in x^{\perp}$ and hence the inner-product $Dp(x)y$ is pointwise 0 $\forall (x, y) \in \sum_X$. If $p \neq |.|$ then there are a measurable number $Dp(x)y$ that have non-zero value and since each term is squared the integral is positive.

3. Symmetry: Symmetry clearly exists.

4. Triangle inequality: Checking this seems non-trivial to me, so HW :).

The three propositions from which the proof of Theorem 1 follows are,

Proposition 1. *There exists a sequence $c_n \to 0$ such that, for any n-dimensional normed space (X, p) there exists an inner-product norm $|.|$ on X with $v(X, p, |.|) \leq c_n$.*

Proposition 2. *For any $(X, p, |.|)$ and integer k with $1 \leq k \leq dimX$, there exists a subspace E of X with dim $E = k$ and $v(E, p\,|_E, |.|\,|_E) \leq v(X, p, |.|)$.*

Proposition 3. *For any k, $\epsilon > 0$ there exists a $\delta > 0$ such that, if dim $E = k$ and $v(E, p, |.|) < \delta$, then (E, p) is the $\epsilon-$Euclidean.*

Lets now prove these propositions. Before we prove Proposition 1 we state the Dvoretzky-Rogers lemma which used in its proof.

Lemma 1. *For every normed space (X,p) with dim $X=n$, $\exists$ an integer $m>\frac{1}{2}\sqrt{n}-1$ and linear operators $T: l_2^n->X$, $U: X->l_\infty^m$ such that $\|T\|=1$, $\|U\|\leq 2$, T is one-to-one , and $UT((x_1,...,x_n))=(x_1,...,x_m)$ for $(x_1,...,x_n)\in l_2^n$.*

Proof of Proposition 1.

Proof. Let the Euclidean norm on X be defined as $|x|=\|T^{-1}(x)\|_{l_2^n}$. If p is differentiable at $x\in X$ and $y\in X$, then $|Dp(x)y|\leq p(y)\leq|y|$. The inequality $p(y)\leq|y|$ holds from the definition of norm of a linear operator and the fact that $\|T\|=1$, .i.e.

$$\begin{aligned}
\|T\| &= sup_x\frac{p(x)}{|x|} \quad \forall x\in X\\
1 &= sup_x\frac{p(x)}{|x|}\\
1 &\geq \frac{p(x)}{|x|}\\
p(x) &\leq |x|
\end{aligned}$$

The inequality $|Dp(x)y|\leq p(y)$ emanates from triangle inequality, which leads to the above set of inequalities. With this we have,

$$\begin{aligned}
q(x) &= \int_{S_X\cap x^\perp}|Dp(x)y|^2 d\lambda_{x^\perp}y\\
&\leq \frac{1}{n-1}sup\{|Dp(x)y|^2 : y\in S_X\cap x^\perp\}\\
&\leq \frac{1}{n-1}
\end{aligned}$$

The last inequality in the above equation is derived from the fact that $|y|=1$ and $|Dp(x)y|\leq|y|$.

Using definition of v() and lemma 1 we have,

$$\begin{aligned}
v(X,p,|.|) &= \int_{S_X} q(x)p(x)^{-2}d\lambda_X(x)\\
&\leq \frac{1}{n-1}\int_{S_X}\|U\|^2\|Ux\|^{-2}d\lambda_X(x)\\
&= \frac{\|U\|^2}{n-1}\int_{S_{l_2^n}}\|UTz\|^{-2}d\lambda_{l_2^n}(z)\\
&\leq \frac{4}{n-1}\int_{S_{l_2^n}}(max_{1\leq i\leq\frac{\sqrt{n}}{2}}|x_i|)^{-2}d\lambda_{l_2^n}(z)\\
&= c_n
\end{aligned}$$

$\square$

The second step in the above derivation, again is from the definition of norm for linear operators. i.e.

$$\begin{aligned}\|U\| &= sup_x \frac{\|Ux\|}{p(x)} \quad \forall x \in X \\ \|U\| &\geq \frac{\|Ux\|}{p(x)} \\ p(x)^{-2} &\leq \|U\|^2 \|Ux\|^{-2}\end{aligned}$$

In third step we take $\|U\|^2$ outside the integral and replace x by Tz where $x \in X$ and $z \in l_2^n$.

In the fourth step we use the inequality $\|U\|^2 \leq 2$ and the definition of l_∞ norm.

Proof of Proposition 2

Proof. Let γ denote normalized rotation invariant measure on the Grassmann manifold Γ of all k-dimensional linear subspaces E of X, then we know that,

$$\int_{\sum_X} f(x,y) d\sigma_X(x,y) = \int_\Gamma d_\gamma(E) \int_{\sum_E} f(x,y) d\sigma_E(x,y)$$

On putting $f(x,y) = (Dp(x)y)^2 p(x)^{-2}$ we see that, $v(X,p,|.|) = \int_\Gamma d_\gamma(E) \int_{\sum_E} f(x,y)d\sigma_E(x,y)$ which is the expected value of $v(E, p\,|_E, |.|\,|_E)$ over all E. Thus there must exists atleast one E for which $v(E, p\,|_E, |.|\,|_E) \leq \int_\Gamma d_\gamma(E) \int_{\sum_E} f(x,y)d\sigma_E(x,y) = v(X,p,|.|)$. □

Proof of Proposition 3

Proof. We prove this result by contradiction. If Proposition 3 was false, then $\exists$ numbers k, ϵ and a sequence (p_n), $n \in \mathcal{N}$ of norms on $E = l_2^k$ such $v(E, p_n, |.|) < \frac{1}{n}$ and p_n fails to be $\epsilon-$Euclidean. Let $S = \{x \in E : |x| = 1\}$. Assume,

$$sup_{x\in S} p_0(x) = 1 > 1 - \epsilon \geq inf_{x \in S} p_0(x)$$

where $p_0(x) = lim_{n->\infty} p_n(x)$. Let,

$$\begin{aligned}A &= \{x \in S : Dp_n(x) \; exists \; for \; n \in \mathcal{N} \cup 0\} \\ B &= \{x \in A : Dp_0(x)y = 0 \; where \; y \in x^\perp\}\end{aligned}$$

Since $p_n's$ are convex, we have $\lambda_E(A) = 1$, $lim_{n->\infty} Dp_n(x) = Dp_0(x)$ where $x \in A$.

By Fatou's lemma,

$\int_{\sum_X} (Dp_0(x)y)^2 d\sigma_E(x,y) = 0$

Thus $\lambda(A \setminus B) = 0$. And so points $x_1, x_2 \in S$ can be connected in S by a rectifiable curve $g(t)$, $a \leq t \leq b$, whose most points are in B. Hence,

$p_0(x_2) - p_0(x_1) = \int_a^b Dp_0(g(t))(g'(t))dt = 0$

which means p_0 is a constant on S which is a contradiction. $\square$

The three propositions have been proved from which the theorem follows. For completeness we state the following lemma.

Lemma 2. *Let $m(n)$ be a sequence of positive integers, such that $m(n) \leq n$ and $lim_{n->\infty} m(n) -> \infty$ and let*

$$\alpha(n) = \frac{1}{n} \int_S (max_{1 \leq i \leq m(n)} |x_i|)^{-2} d\lambda(x)$$

where λ is a normalized rotation invariant measure on the unit sphere S of l_2^n. Then $lim_{n->\infty} \alpha(n) = 0$