Geometric Constraint Lecture(Mar 21-30)

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1 Problem Categories

2 Five Questions

1. Given graph G, characterize d for which (G, d) has a realization. Here d are constraints, for example distance constraints.

2. Given graph G and constraints d, provide a realization.

3. Given grpah G, generically classify it into two categories:

- It has finite number of realizations.
  - One realization
  - Many realizations
- It has infinite number of realizations.

4. Given G, generically characterize the realization space.

5. Given nongeneric G, with fixed or restricted d, answer question 3 and 4. Give the classification and description of its realization space.

3 Working on these Five Questions

3.1 Question 1

Problem: **G is a complete distance graph, find** $\{d : (G, d)\ \textit{has a realization in}\ \mathbb{R}^k \textit{space}\}$.

Theorem: Cayley-Menger conditions are the necessary and sufficient conditions that (G, d) has a realization in $\mathbb{R}^k$space.

3.2 Question 4

Question 1 and 4 are equivalent in the sense that if we understand one of them, we understand the other.

3.3 Question 3

3.3.1 Laman's theorem: A graph G generically has only finitely many solutions iff the following two conditions hold:

1. $\forall subgraph\ S \subseteq G,\ 2|V_S| - |E_S| \geq 3$
2. $2|V_G| - |E_G| = 3$

3.3.1.1 General Laman theorem: A graph $G = (V, E)$ generically has at most finitely many solutions iff $\exists subgraph\ G' = (V, E')\ with\ E' \subseteq E$ such that

1. $\forall subgraph\ S \subseteq G',\ 2|V_S| - |E_S| \geq 3$
2. $2|V_{G'}| - |E_{G'}| = 3$

3.3.1.2 Definition of Generic

Embedding: can be understanded in the following three ways:

- $(x_1y_1x_2y_2 \cdots x_ny_n) \subseteq \mathbb{R}^{2n}$
- $d_{\bar{G}} \subseteq \mathbb{R}^{|V_{\bar{G}}|}$
- $\mathbb{R}^{2n} \diagdown E_2$

Given (G, d) **o/w** d_G
$\{(x_1y_1x_2y_2 \cdots x_ny_n) : (x_v - x_w)^2 + (y_v - y_w)^2 = d_{vw}^2\ \forall (v, w) \in E(G) \diagdown E_2\}$
$\leftrightarrow$
$d_{\bar{G}} = \{(x_a - x_b)^2 + (y_a - y_b)^2, \cdots (a, b) \notin E(G), : (x_v - x_w)^2 + (y_v - y_w)^2 = d_{vw}^2\ \forall (v, w) \in E(G)\}$

There is one to one map between these two sets.

Typical embedding: An embedding Q of G is generic if

$\exists a\ small\ enough\ neighborhood\ of\ \ (d_{Q,G}),\ \ d_{Q,G\pm\xi},\ \ all\ realization\ of\ \ (G, d_{Q,G\pm\xi})\ are\ rigid \Leftrightarrow Q\ \ is\ \ rigid$**.**

Alternately, One can define a small enough neighborhood of Q itself, Require all corresponding realization to be rifid for their corresponding distance values.

3.3.1.3 Rigidity of Graph:

Def1: A Graph is rigid $\exists a\ \ generic\ \ embedding\ \ that\ \ is\ \ rigid.$

Def2: A Graph is rigid if $all\ \ generic\ \ embeddings\ \ are\ \ rigid.$

These two definitions turn out to be equivalent and we will give the proof in the following notes.

Def3: A Graph is globally rigid if $it\ \ is\ \ rigid$ &

$\forall generic\ embedding\ Q\ with\ distances\ d_{G,Q},\ Q\ is\ the\ unique\ generic\ realization\ of\ (G, d_{G,Q})$

Def4: An embedding Q of G is rigid if

$\exists\ a\ \ small\ \ enough\ \ \mathbb{R}^{2n} - neighborhood\ \ Q_\xi,\ \ such\ \ that\ \ for\ \ \forall Q' \in Q_\xi,\ \ Q'\ \ is\ \ a\ \ realization\ \ of\ (G, d_{Q,G}) \Leftrightarrow Q'\ \ is\ \ a\ \ rigid\ \ motion\ \ of\ \ Q.$

Def5: An embedding Q of G is generic if

$\exists\ a\ small\ enough\ \mathbb{R}^{2n} - neighborhood\ Q_\xi,\ such\ that\ \forall Q' \in Q_\xi,\ Q'\ is\ rigid \Leftrightarrow Q\ \ is\ \ rigid.$

Question: Given a particular embedding $Q \in \mathbb{R}^{2n}\ \ of\ \ G$, Decide

1. is Q generic?
2. is Q rigid?

3.3.1.4 Laman's theorem Proof

Laman's-theorem Given graph $G = (V, E)$, G is generically rigid in $2[k]$ dimensions $\Leftrightarrow$ $[\Rightarrow]$ $\exists\, G' = (V, E'),\ \ E' \subseteq E$ such that

1. $\forall S \subseteq G',\ \ 2\,|V_S| - |E_S| \geq 3 \quad [k\,|V_S| - |E_S| \geq \begin{pmatrix} k+1 \\ 2 \end{pmatrix}]$
2. $2\,|V| - |E'| = 3 \quad [k\,|V| - |E'| = \begin{pmatrix} k+1 \\ 2 \end{pmatrix}]$

- $\Rightarrow$Proof:

Def: Rigidity Matrix $R_{P,G}$ of an embedding P in $\mathbb{R}^k$of a graph $G = (V, E)$ is

$$\begin{pmatrix} & p_1 & p_2 & \underbrace{}_{k} & p_i & p_{|v|} \\ e_1 & & & & & \\ e_2 & & \ddots & & & \\ \vdots & & & & & \\ e_i & 0 & 0 & 0 & (p_i - p_j) & 0 \\ & & & & & \cdots \end{pmatrix} \quad e_i = (v_i - v_j)$$

Fact1: Consider the vector $\begin{pmatrix} u_1 \\ \vdots \\ u_{|V|} \end{pmatrix}$ $\quad u_i \in \mathbb{R}^k,\ \ s.t\ \ R_{P,G}U = 0$ it means

$$u_1 \cdots u_{|V|} \quad s.t \quad \forall i, j\ \ (v_i, v_j) \in E \quad < (p_i - p_j)(u_i - u_j) >= 0$$

The set of independent U's with this property is a basis for the tangent space of the variety $\forall (v_i, v_j) \in E \quad \|q_i - q_j\| - \|p_i - p_j\| = 0\ \ at\ the\ point\ \ p \in \mathbb{R}^{k|V|}$

Fact2: row rank of $R_{P,G} = k\,|V| - rank\ \ of\ \ S_{P,G}$

Lemma1: P is generic then P is rigid$\Leftrightarrow R_{P,G}\ \ has\ \ rank\ \ k\,|V| - \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$

Proof:

$\Leftarrow$True, Full rank then rigid

$\Rightarrow if \quad R_{P,G} \quad has \quad rank \; < k\,|V| - \begin{pmatrix} k+1 \\ 2 \end{pmatrix} \quad then \quad if \quad P \quad is \quad rigid \Rightarrow P \; is \; not \; generic.$

Lemma2: if P is generic embedding of $G = (V, E)$ & $R_{P,G}$ has $(k\,|v| - \begin{pmatrix} k+1 \\ 2 \end{pmatrix}$ rank then

Laman's theorem for general dimensions k must hold.

Laman's theorem $\Rightarrow$for general k dimensions follows from lemma1, lemma2 and definition of generic rigidity of G

- $\Leftarrow$ Proof

Laman's theorem $\Leftarrow$only for 2dimensions, starting from a graph $G = (V, E)$ for which the RHS holds, laman showed

1. G has "Henneberg" construction
2. if G has a henneberg construction, then it has a generic and rigid embedding.

Laman's theorem $\Leftarrow$: If a graph $G = (V, E)$ satisfies Laman condition, then $\exists$a generic rigid embedding $P \in \mathbb{R}^{2|V|}$ of G

Lemma1: If G satisfies Laman condition 1, then graph G has a Henneberg construction

Observation1: If graph G has a henneberg construction, then the abstract underlying matroid underlying G is "1-extendible"

Lemma2: If G satisfies the Laman condition 1 and the abstract rigidity matroid underlying G is "1-extendible" then the edges in G are generically independent

Observation2: If G satisfies both Laman conditions, then G has a generic rigid embedding.

Matroid:

Def1: Matroid M: finite set $(E, \mathcal{I})$, & $\mathcal{I} \subseteq \mathcal{P}ower(E)$ &

1. $\phi \in \mathcal{I}$

2. $S \in \mathcal{I}$ $then$ $\forall Q \subseteq S,\ \ Q \in \mathcal{I}$

3. $S(not\ \ maximal) \in \mathcal{I},\ \ then\ \exists u \notin S,\ \ \forall v \in S,\ \ s.t\ \ (S \diagdown v \cup \{u\}) \in \mathcal{I}$

(3' If $S_1, S_2 \in \mathcal{I},\ \ |S_1| < |S_2|\ \ then\ \exists an\ \ element\ \ u \in S_2 \diagdown S_1,\ \ s.t\ \ S_1 \cup \{u\} \in \mathcal{I}$)

Examples:

1. Let E be a set of vectors in $\mathbb{R}^m$ or $\mathcal{F}^m$(any field), Let $\mathcal{I}$ be the set of all linearly independent subset of E, claim $(E, \mathcal{I})$ is a matroid.

2. $G = (V, E),\ M_G = (E, \mathcal{I}),\ \mathcal{I} = \{any\ subset\ of\ edge\ that\ does\ not\ have\ a\ cycle\}$

3. Fano plane $E = \{p_1 \cdots p_7\},\ \mathcal{I} = set\ of\ all\ subsets\ of\ points\ that\ are\ not\ collinear$

Facts: All maximal[1] independent sets have the same number elements called $rank(M)$ in them

Def2: Matroid M: finite set E & a closure $<>$: $\mathcal{P}ower(E) \rightarrow \mathcal{P}ower(E)$

1. $Q \subseteq < Q >$

2. $<< Q >>=< Q >$

3. If $Q_1 \subseteq Q_2$ then $< Q_1 > \subseteq < Q_2 >$

4. If $s, t \in E \diagdown < T >$ then $s \in < T \cup \{t\} > \Leftrightarrow t \in < T \cup \{s\} >$

Examples:

1. fano plane, $E = \{p_1 \cdots p_7\}$,$< p_1 >= p_1$

for $k \geq 2$, if $p_1 \cdots p_7$ lie on a line, then $< p_1 \cdots p_7 >$ are all points on this line

if $p_1 \cdots p_7$ donot lie on any line, then $< p_1 \cdots p_7 >$ are all points

2. affine dependent matroid

$E = \{p_1 \cdots p_7\} \in \mathbb{R}^m$, $< p_1 \cdots p_7 >$=all points in the affine span of $(p_1 \cdots p_7) \in E$

Def3: Infinitisemal complete rigidity matroid

Given $P = \{p_1 \cdots p_m\},\ \ p_i \in \mathbb{R}^k$, embedding of G

$M = (E, \mathcal{I})$ $\mathcal{I}$ =**set of all linearly independent subsets of rows of the complete rigidity matrix**

[1]An independent set to which no elements can be added & maintain independent

E =**rows of the complete rigidity matrix**

Def4: Generic k-dimension rigidity matroid of a graph $G = (V, E)$ $\mathcal{G}_k = (E, \mathcal{I})$, $\mathcal{I}$ is obtained as follows:

take any generic embedding P of G in k-dimension & define $\mathcal{I}$ to be the same as for the infinitesimal rigidity matroid for P_G

Def5: Abstract rigidity matroid on a complete graph of a vertex set V

E =**set of all possible edges of** V

The closure operator satisfies all matroid conditions and the following condition:

5. if $Q_1, Q_2 \subseteq E$, **&** $|V(Q_1)| \cap |V(Q_2)| < k$ **, then** $< Q_1 \cup Q_2 > \subseteq Completion(V(Q_1)) \cup Completion(V(Q_2))$

Def6: If $Q_1, Q_2 \subseteq E$ are abstract rigid[2], and $|V(Q_1) \cap V(Q_2)| \geq k$, then $< Q_1 \cup Q_2 >$ is rigid

3.3.2 Jackson-Jordon theorem

for $d = 2$, $G = (V, E)$ **generically has an unique solution (globally rigid)** $\Leftrightarrow G$ **is redundantly rigid &** 3**-connected or it is a triangle.**

Redundantly rigid: removal of any edge preserves rigidity of G.

3.3.2.1 Hendrickson's theorem

$G = (V, E)$ **is globally rigid in** d **dimension** $\Leftrightarrow G$ **is reduntantly rigid for** d **dimension &** $(d+1)$ **connected.**

$\Leftarrow$**proved**

$\Rightarrow$**Conely disproved for** $d \geq 3$

$\Rightarrow$**proved (Jackson-Jordon theorem)**

[2] Q is abstract rigid if $< Q >= Completion(V(Q))$

3.3.3 Owen's theorem

A graph is quadratically solvable $\Leftrightarrow$it is not 3-connected.

Quadratically solving: A consraint system (G, d) is quadratically solvable if it is triangularizable into quadratics.

$\Leftarrow$proved

$\Rightarrow$For planar graph, a graph is quadratically solvable $\Rightarrow$it is not 3-connected

$\Rightarrow$For general graph, open problem