Geometric Constraint Lecture(Apr 26)

May 3, 2006

1 Owen's theorem:

If the underlying abstract graph of the configuration is 3-connected or if any of the graphs in a certain recursive decomposition of the graph is 3-connected then the configuration is not RS (thus not QS), otherwise it is QS.

1.1 No 3-connected subgraphs => QS

The Algorithm to compute a graph to see if it is QS:

Input: graph for a well-constrained system

Algorithm:

The graph for a well-constrained system is 2-connected, otherwise it cannot be well-constrained. A 2-connected graph can be

(1) containing no 3-connected subgraphs

or

(2) containing 3-connected subgraphs

In the first case, it can be split into edges and triangles.

The algorithms splits 2-connected graph into components and repeated the splitting for the split components and until no more splits are possible. And if there are some 3-connected components in the end, then it is not QS, otherwise it is QS.

1.2 3-connected graph => not QS

1.2.1 Three equivalent statements:

1. The geometry can be constructed in principle to satisfy the dimensions using a straight edge and compasses and ruler with spacing equal to the distance dimension values and the cosine of the angle dimension values.
2. The configuration can in principle be solved algebraically using only arithmetic operations plus square root.

3. The coordinates for the geometries all lie in a normal field extension over the dimension values of degree 2^n for some n.

1.2.2 Theorem: The constraints equations represented by a 3-connected graph do not have solutions in a normal field extension of degree 2^n over the dimension values for any finite n.

If we can prove the above theorem, then since those three statements are equivalent, then it is not QS.

Proof: proved by contradiction.

Assume that the theorem is false, then none of the geometries can have both coordinates in a field extension of degree smaller than 2^n which is absurd.

Lemma: If the solutions to the constraint equations represented by a 3-connected graph have coordinates in a normal field extension of degree 2^n then none of the geometries can have both coordinates in a smaller field.

Proof of Lemma: We can decomposed a normal field extension of degree 2^n into a sequence of field extensions each of degree 2 over its predecessor. Write it as F0F1..Fn, so F0 is the field spanned by the distance constrained.

Supposed that at least one geometry lies in Fn-1 and the rest lie in Fn but not in Fn-1. The geometries that lie in Fn but not in Fn-1 can be written as

g_j=u_j+yv_j, g_j not in Fn-1. so g_j=u_j-yv_j, g_j not in Fn-1 is another solution, and they are reflected about a line perpendicular to v_j, this line must be the same for all the g_j not in Fn-1 and are connected by a constraint equation because the reflection must conserve the value of the constraints specified. Similiarly, any objects in Fn-1 which are connected to objects not in Fn-1 must lie on this lines. Because the dimension values are independent, then at most two objects can be on the same line. So these two points can separated the graph into two parts and it is not 3-connected. This is in contradict with we assumed.

2 My comments:

- About the second part of the proof:

The proof given in the second part is not totally correct as the arthur stated. But to be honest, I do not know exactly which step it goes wrong. I doubt it might be the place where they claim there is a single line for all the objects in Fn but not in Fn-1. I do not understand that part clearly, in the paper, the arthur mentioned something about Galois group which is a different view to look at this reflection line. It might help to try to understand it from that way.

- About the paper of Owen: Are all 3-connected Generic Constraint Configurations of Points in a Plane Non-radical?

This paper gave the outline of proof for planar graph and non-planar graph. Again, I think they use some graph operations trying to decompose into a subgraph then by looking at the properties of these subgraphs, we can know whether the problem is RS or not. It involves a lot of other graph properties, I need some more time or knowledge to understand them. Basically, I think they proved the theorem for planar graph, but for general graph, more cases analysis is needed.