2.4. Infinitesimal and Generic Rigidity. For $E \subseteq K$, we define the *dependency number* of E to be the number of independent relations among the rows of $R(\mathbf{p})$ corresponding to E or, equivalently, as the dimension of $\mathcal{S}(E)$; we denote it by $dn(E)$. We define the *degree of freedom* of E to be the dimension of $\mathcal{V}(E)$ minus the dimension of $\mathcal{D}(E)$ and denote it by $\mathrm{df}(E)$.

THEOREM 2.4.1. *Let V, the general embedding* $\mathbf{p}$ *of V into $\mathbb{R}^m$ and $E \subseteq K$ be given.*

a. $\mathrm{dn}(E) \geq 0$, *with equality if and only if E is independent.*
b. $\mathrm{df}(E) \geq 0$, *with equality if and only if E is rigid.*
c. *If* $|V(E)| \geq m$, *then* $|E| = m|V(E)| - \binom{m+1}{2} + \mathrm{dn}(E) - \mathrm{df}(E)$.
d. *If* $|V(E)| \leq (m+1)$, *then* $\mathrm{dn}(E) = 0$, *(i.e. E is independent) and* $\mathrm{df}(E) = \binom{|V(E)|}{2} - |E|$.

PROOF. Parts a and b follow at once from the definitions of $\mathrm{dn}(E)$ and $\mathrm{df}(E)$.

Continued in the next page …

Assume $|V(E)| = n \geq m$. By the corollaries to Theorems 2.3.1 and 2.3.2, we have:

$$\begin{aligned}
\mathrm{df}(E) &= \dim(\mathcal{V}(E)) - \dim(\mathcal{D}(E)) \\
&= [nm - |E| + \mathrm{dn}(E)] - \left[m(n - |V(E)|) + \binom{m+1}{2}\right] \\
&= m|V(E)| - |E| - \binom{m+1}{2} + \mathrm{dn}(E).
\end{aligned}$$

Now assume $|V(E)| \leq m+1$. By Lemma 2.2.1c, E is independent and $\dim(\mathcal{S}) = 0$. Thus, by Corollary 1 to Theorem 2.3.1, $\dim(\mathcal{V}(E)) = nm - |E|$. Similarly, $\dim(\mathcal{D}(E)) = \dim(\mathcal{V}(K(V(E))) = mn - \binom{n}{2}$ and part d follows by direct computation. □

Theorem 2.4.1 can be used to produce most of the standard theoretical results about infinitesimal rigidity. However, we will prove these results later in a more general setting. Hence, we will not pursue this investigation beyond using this result to prove the rigidity result parallel to Theorem 2.2.1 and to demonstrate, with the following exercises, the relationship between stresses and infinitesimal motions in working with specific examples.

COROLLARY 2.4.1. *If a framework $(V(E), E, \mathbf{p})$ is infinitesimally rigid for some general embedding $\mathbf{p}$ of V into $\mathbb{R}^m$ then $(V(E), E, \mathbf{q})$ is infinitesimally rigid for all generic embeddings $\mathbf{q}$ of V into $\mathbb{R}^m$.*

PROOF. Let $\mathrm{df}_{\mathbf{p}}(E)$, $\mathrm{df}_{\mathbf{q}}(E)$, $\mathrm{dn}_{\mathbf{p}}(E)$ and $\mathrm{dn}_{\mathbf{q}}(E)$ denote the degree of freedom and dependency number of E with respect to the embeddings $\mathbf{p}$ and $\mathbf{q}$, where $\mathbf{p}$ is any general embedding of V into $\mathbb{R}^m$ and $\mathbf{q}$ is any generic embedding of V into $\mathbb{R}^m$. We wish to show that $\mathrm{df}_{\mathbf{p}}(E) = 0$ implies that $\mathrm{df}_{\mathbf{q}}(E) = 0$. If $|V(E)| < m$, we have from Theorem 2.4.1d: $\mathrm{df}_{\mathbf{q}}(E) = |V(E)|(|V(E)| - 1)/2 - |E| = \mathrm{df}_{\mathbf{p}}(E)$. Now assume that $|V(E)| > m$. Similarly, from Theorem 2.4.1c we get that

$$\mathrm{dn}_{\mathbf{p}}(E) - \mathrm{df}_{\mathbf{p}}(E) = \mathrm{dn}_{\mathbf{q}}(E) - \mathrm{df}_{\mathbf{q}}(E).$$

Since $\mathbf{q}$ is a generic embedding, $\mathrm{dn}_{\mathbf{p}}(E) \geq \mathrm{dn}_{\mathbf{q}}(E)$ (see Exercise 2.25 below). Hence, $\mathrm{df}_{\mathbf{p}}(E) \geq \mathrm{df}_{\mathbf{q}}(E) \geq 0$. □

This result leads us to define an edge set E to be *generically rigid* (for dimension m) if it is rigid with respect to all generic embeddings into m-space and to reformulate the corollary:

COROLLARY 2.4.2. *If a framework $(V(E), E, \mathbf{p})$ is infinitesimally rigid for some general embedding $\mathbf{p}$ of V into $\mathbb{R}^m$ then E is generically rigid for dimension m.*

2.6. Isostatic Sets. In the next chapter, we will prove that any matroid is uniquely determined by its collection of independent sets or its rank function. We will also show that the abstract rigidity matroids are uniquely determined by the collection of those independent sets which are rigid. Such edge sets are called *isostatic*. In the next lemma we state some of the useful facts about isostatic sets.

LEMMA 2.6.1. *Let V be a given finite set, let $E \subseteq K = K(V)$ and consider an m-dimensional abstract rigidity matroid for V.*

a. *Assume that $|V(E)| \leq (m+1)$. Then E is independent; furthermore, E is rigid, and hence isostatic, if and only if $E = K(V(E))$.*

Continued in the next page …

b. *Assume that* $|V(E)| \geq (m+1)$ *and that* E *is isostatic. Then:*
  (i) $|E| = m|V(E)| - \binom{m+1}{2}$;
  (ii) *Each vertex of* $(V(E), E)$ *has valence at least* m;
  (iii) $(V(E), E)$ *has a vertex with valence less than* $2m$.

c. *Assume that* $|V(E)| \geq (m+1)$. *Then, if any two of the following conditions hold, all three hold and* E *is isostatic.*
  (i) $|E| = m|V(E)| - \binom{m+1}{2}$;
  (ii) E *is independent;*
  (iii) E *is rigid.*

PROOF. Part a: Let $|V(E)| \leq (m+1)$. By Lemma 2.5.6b, $r(K(V(E)) = |K(V(E))|$. By Lemma 2.5.5a $K(V(E))$ is independent; then by, Lemma 2.5.5c, E is independent. For any $(i,j) \in (K(V(E)) - E)$, $E \cup \{(i,j)\}$ is also independent and, hence, $(i,j) \notin \langle E \rangle$. We conclude that $\langle E \rangle = E$. It follows at once that E is rigid, and hence isostatic, if and only if $E = K(V(E))$.

Part b: Now let $|V(E)|$ be greater than m and assume that E is isostatic. Since E is independent, $|E| = r(E)$ (Lemma 2.5.5a). Since E is rigid, $r(\langle E \rangle) = r(K(V(E))) = m|V(E)| - \binom{m+1}{2}$ (Lemma 2.5.6b). By Lemma 2.5.5f, $r(E) = r(\langle E \rangle)$. Combining these inequalities gives $|E| = m|V(E)| - \binom{m+1}{2}$.

Suppose that $(V(E), E)$ contained a vertex i of valence less than m. Let $U = V(E) - \{i\}$; let $F = E(U)$; let $H = E - F$. We have $|H| < m$ and $|U| = |V(E)| - 1$ and $V(F) \subseteq U$. We have by Theorem 2.5.4 that $|F| \leq m|V(F)| - \binom{m+1}{2}$. Combining this and the fact that $V(F) \subseteq U$, we have:

$$\begin{aligned}
|E| = |F| + |H| &\leq [m|V(F)| - \binom{m+1}{2}] + (m-1) \\
&\leq [m|U| - \binom{m+1}{2}] + (m-1) \\
&= [m(|V(E)| - 1) - \binom{m+1}{2}] + (m-1) \\
&= [m|V(E)| - \binom{m+1}{2}] - 1, \text{ contradiction!}
\end{aligned}$$

Thus E every vertex in E has valence at least m.

Finally, we note that the sum of the valences of the vertices in $(V(E), E)$ is $2|E|$. Hence, the average valence is:

$$2\frac{|E|}{|V(E)|} = 2\frac{m|V(E)| - \binom{m+1}{2}}{|V(E)|} = 2m - \frac{\binom{m+1}{2}}{|V(E)|}.$$

Thus, the average valence is less than $2m$ and there must be a vertex of valence less than $2m$.

The proof of part c is left as an exercise for the reader. □

THEOREM 2.6.1. *Let the finite set V and the positive integer m be given and let $\mathcal{A}_m$ be an m-dimensional abstract rigidity matroid for V. Let $F \subseteq K = K(V)$ be an isostatic set in $\mathcal{A}_m$ and let E be a 0-extension of F with $V(E) \subseteq V$. Then E is isostatic in $\mathcal{A}_m$. Conversely, if E is isostatic in $\mathcal{A}_m$ and $(V(E), E)$ has a vertex of valence m, then E is a 0-extension of some isostatic set in $\mathcal{A}_m$.*

PROOF. Let U be an m-subset of $V(F)$, $i \in (V - V(F))$ and $E = F \cup \{(i,j) | j \in U\}$. Assume that F is isostatic. Since $|V(F)| \geq m$, we have, by Lemma 2.5.6b,

$$|F| = r(F) = m|V(F)| - \binom{m+1}{2}.$$

By Lemma 2.5.6,

$$r(E) = r(F) + m = m(|V(F)| + 1) - \binom{m+1}{2} = m|V(E)| - \binom{m+1}{2}.$$

By direct computation,

$$|E| = |F| + m = m(|V(F)| + 1) - \binom{m+1}{2} = m|V(E)| - \binom{m+1}{2}.$$

By Lemma 2.5.5a, E is independent and then, by Lemma 2.6.1c, E is isostatic.

Continued in the next page …

Conversely, if E is isostatic, then F as a subset of E is independent. But then, by a direct count and Lemma 2.6.1c, we conclude that F is isostatic. □

COROLLARY 2.6.1. *Let the finite set V and the positive integer m be given and let $\mathcal{A}_m$ be an m-dimensional abstract rigidity matroid for V. Let $F \subseteq K = K(V)$ be independent in $\mathcal{A}_m$ and let E be a 0-extension of F with $V(E) \subseteq V$. Then E is independent in $\mathcal{A}_m$. Conversely, if E is independent in $\mathcal{A}_m$ and $(V(E), E)$ has a vertex of valence m, then E is a 0-extension of some independent set in $\mathcal{A}_m$.*

COROLLARY 2.6.2. *Let the positive integers m and h and the finite set V, with $|V| \geq m + h$, be given and consider $\mathcal{A}_m$, an abstract rigidity matroid for V. Let U_0 be an m-subset of V, let $i_1, \ldots, i_h$ be distinct vertices in $V - U_0$. Let $E_0 = K(U_0)$ and, for $j = 1, \ldots, h$, let E_j be a 0-extension of E_{j-1}. Then, for $j = 1, \ldots, h$, E_j is isostatic in $\mathcal{A}_m$.*

THEOREM 3.11.3 (SEE LEMMA 2.6.1). *Let V be a finite set, let $E \subseteq K = K(V)$ and consider an m-dimensional abstract rigidity matroid for V.*

a. *Assume that $|V(E)| \leq (m+1)$. Then E is independent; furthermore, E is rigid, and hence isostatic, if and only if $E = K(V(E))$.*

b. *Assume that $|V(E)| \geq (m+1)$ and that E is isostatic. Then:*
   (i) $|E| = m|V(E)| - \binom{m+1}{2}$;
   (ii) *Each vertex of $(V(E), E)$ has valence at least m;*
   (iii) *$(V(E), E)$ has a vertex with valence less than $2m$.*

c. *Assume that $|V(E)| \geq (m+1)$. Then, if any two of the following conditions hold, all three hold and E is isostatic.*
   (i) $|E| = m|V(E)| - \binom{m+1}{2}$;
   (ii) *E is independent;*
   (iii) *E is rigid.*

This theorem has a useful corollary not included in Chapter 2, but included here.

COROLLARY 3.11.1. *Let the finite set V be given, let $E \subseteq K = K(V)$, and consider an m-dimensional abstract rigidity matroid for V. Then E is rigid if and only if either $|V(E)| \leq m+1$ and $E = K(V(E))$ or $|V(E)| \geq m+1$ and $r(E) = m|V(E)| - \binom{m}{2}$.*

THEOREM 4.1.1. *Let $\mathcal{A}_2$ be a 2-dimensional abstract rigidity matroid on n vertices. The following are equivalent:*

(i) $\mathcal{A}_2 = \mathcal{G}_2(n)$;

(ii) $\mathcal{A}_2$ *has the* 1*-extendability property;*

(iii) *The independent sets of $\mathcal{A}_2$ are those sets which satisfy Laman's condition;*

(iv) *All cycles of $\mathcal{A}_2$ are rigid;*

(v) *For any closed set E of $\mathcal{A}_2$ with cliques $E_1, \ldots, E_k$, $r(E) = r(E_1) + \cdots + r(E_k)$;*

(vi) $r(E) = \min \sum_{i=1}^{k}(2|V(E_i)| - 3)$, *where the minimum is taken over all collections $\{E_i\}$ of nonempty sets such that $E = \cup E_i$.*

While developing this theory of planar rigidity, we will also consider which results about graph connectivity have analogous formulations in terms of rigidity in dimension 2.

In the closing section of Chapter 3, we listed all of the results produced to date about abstract rigidity matroids. Here we summarize those results as they apply in dimension two. It is left to the reader to verify the accuracy of this summary and to fill in a few missing, but elementary proofs.

4.2. Combinatorial Characterizations of $\mathcal{G}_2(n)$. In Exercise 4.3 above we saw that the generic rigidity matroid is not the only 2-dimensional abstract rigidity matroid. Our goal is to characterize generic rigidity in the plane and our first step is to relate 1-extendability to Laman's condition.

LEMMA 4.2.1. *Let V be a finite set and let $\mathcal{A}_2$ be any 2-dimensional abstract rigidity matroid for V that satisfies the 1-extendability property. Then $E \subseteq K$ is independent in $\mathcal{A}_2$ if and only if E satisfies Laman's condition for dimension 2.*

PROOF. By Theorem 2.5.4 we need only prove that every edge set E which satisfies Laman's condition for dimension 2 is independent in $\mathcal{A}_2$. We give a proof by contradiction. Suppose that E is a vertex and edge minimal dependent set which satisfies Laman's condition. By minimality, E has no vertex of valence 1 or 2, so E has a vertex v of valence 3. Let $F = E - v$ and denote the set of neighbors of v by $N = \{x, y, z\}$. We have $K(N) \not\subseteq F$, since otherwise $K(N + v) \subseteq E$ and tetrahedra violate Laman's condition. Moreover, $K(N) \subseteq \langle F \rangle$, since otherwise we have $F + (x, y)$, say, is independent and E is a 1-extension of $F + (x, y)$ and so is also independent. For $e \in K(N)$, define $X_e \subseteq F$ to be the minimal set such that $e \in \langle X_e \rangle$. If $X_e \neq e$, then $e \notin X_e$ and $X_e + e$ is a minimal set violating Laman's Condition. So $|X_e| = 2|V(X_e)| - 3$, hence $\langle X_e \rangle = K(V(X_e))$. Thus $\langle X_{e_1} \cup X_{e_2} \cup X_{e_3} \rangle = \langle X_{e_1} \cup X_{e_2} \cup X_{e_3} \cup \{e_1, e_2, e_3\} \rangle = K(V(X_{e_1} \cup X_{e_2} \cup X_{e_3}))$ by axiom **C6**. We conclude that $(X_{e_1} \cup X_{e_2} \cup X_{e_3})$ is rigid. But, $X_{e_1} \cup X_{e_2} \cup X_{e_3}$ is also independent. Hence, we have $|X_{e_1} \cup X_{e_2} \cup X_{e_3}| = 2|V(X_{e_1} \cup X_{e_2} \cup X_{e_3})| - 3$. But then it follows that $(X_{e_1} \cup X_{e_2} \cup X_{e_3}) + v$ violates Laman's Condition, a contradiction. □

The preceding lemma serves to characterize $\mathcal{G}_2(n)$ combinatorially, and the following theorems are immediate consequences.

THEOREM 4.2.1 (LAMAN'S THEOREM). *Let V be a finite set and let $\mathcal{G}_2(n)$ be the 2-dimensional generic rigidity matroid for V. Then $E \subseteq K$ is independent (isostatic) in $\mathcal{G}_2(n)$ if and only if E satisfies Laman's condition for dimension 2 (and $|E| = 2|V(E)| - 3$).*

THEOREM 4.2.2. *$\mathcal{G}_2(n)$ is the unique maximal 2-dimensional abstract rigidity matroid on n vertices.*

THEOREM 4.2.3. *Let V be a finite set and let $\mathcal{A}_2$ be any 2-dimensional abstract rigidity matroid for V. Then $\mathcal{A}_2$ is the 2-dimensional generic rigidity matroid for V if and only if $\mathcal{A}_2$ satisfies the 1-extendability condition.*