Geometric Constraint Lecture(Mar 6-8)

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1 Problem Categories

Type One

- Fixed Dimension
- Partially Metric Space
- Exact Realization (usually no distortion)
- Generic or Nongeneric
- Embedding in Euclidian/Projective/Informal Hyperbolic Space
- General or Specail (especially for nongeneric)
- Special Regular Input or General Input
- Combinatorial or Algebraic

Type Two

- Min Dimension
- Complete Metric Space
- Distortion Allowed
- Embedding in LP Space or other Metric Space
- Symmetric Input or General Input
- Combinatorial or Analytic

2 Five Questions

1. Given graph G, characterize d for which (G, d) has a realization. Here d are constraints, for example distance constraints.

2. Given graph G and constraints d, provide a realization.

3. Given grpah G, generically classify it into two categories:

- It has finite number of realizations.
  - One realization
  - Many realizations
- It has infinite number of realizations.

4. Given G, generically characterize the realization space.

5. Given nongeneric G, with fixed or restricted d, answer question 3 and 4. Give the classification and description of its realization space.

3 Working on these Five Questions

3.1 Question 1

Problem: G is a complete distance graph, find $\{d : (G, d) \textit{ has a realization in } \mathbb{R}^k \textit{space}\}$.

Theorem: Cayley-Menger conditions are the necessary and sufficient conditions that (G, d) has a realization in $\mathbb{R}^k$space.

Proof:

3.1.1 Cayley-Menger conditions are the necessary conditions for (G, d) has a realization in $\mathbb{R}^k$space.

First let's look at the following fact.

The following $(n+1) \times (n+1)$ matrix is a distance matrix in $\mathbb{R}^k$space,

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & d_{12} & d_{13} & \cdots & d_{1n} \\ 1 & d_{21} & 0 & d_{23} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{n1} & d_{n2} & \cdots & \cdots & 0 \end{bmatrix}$$ **,volume of $k+2$ simplex is 0.**

The above statement is equivalent to the following formula:

Cayley-Condition: The volume of $k+2$ simplex is 0.

$$\forall_{(k+2) Simplex\ spanned\ by (P_1 \cdots P_{k+2})} Vol_{k+2}(P_1 \cdots P_{k+2}) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & d_{12} & d_{13} & \cdots & d_{1\,k+2} \\ 1 & d_{21} & 0 & d_{23} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{k+2\,1} & d_{k+2\,2} & \cdots & \cdots & 0 \end{bmatrix} = 0$$

Menger-Condition: The volume of smaller or equal to $k+1$ simplex is not negetive.

$$\forall_{j \le k+1} Vol_{j\ simplex}(P_1 \cdots P_j) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & d_{12} & d_{13} & \cdots & d_{1j} \\ 1 & d_{21} & 0 & d_{23} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{j1} & d_{j2} & \cdots & \cdots & 0 \end{bmatrix} \ge 0$$

Cayley-Menger conditions are the necessary conditions for d such that (G, d) has a realization in $\mathbb{R}^k$ space. [1]

3.1.2 Cayley-Menger conditions are also the sufficient condition for (G, d) has a realization in $\mathbb{R}^k$ space.

Let's first look at the following two definitions:

Take $n+1$ points $v_0 \cdots v_n$ in a metric[2] space, put one point v_0 at the origin.

Def 1. Gram matrix $G_{ij} =< v_i, v_j > for\ i, j > 0$.

Def 2. Metric matrix $M_{ij} = \frac{1}{2}(d_{0i}^2 + d_{0j}^2 - d_{ij}^2)$.

Theorem

A set of $n+1$ points in a metric space is realized in $\mathbb{R}^k$ space iff the metric matrix M_{ij} is symmetric; has positive eigenvalues and has rank $\le k$.

$\Rightarrow$Proof: If $n+1$ points lie in $\mathbb{R}^k$, let their coordinate $k \times n$ matrix $\begin{pmatrix} v_1 & \cdots & v_n \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix} = V$. Then $G = V^T V$ turns out to be exactly the metric matrix M.

[1] Cayley condition itself is the necessary condition that (G, d) is realizable in $\mathbb{C}^k$space.

[2] it satisfies Cayley-Menger conditions, positive distances and triangle inequality.

We can verify that G is symmetric and has positive eigen values from its definition, and G has rank at most k because V has rank of k.

$\Leftarrow$Proof: We will take a metric matrix of $n+1$ points in some metric space and using the fact that it is symmetric, has positive eigen values and rank k to generate a realization in $\mathbb{R}^k$ of M.

Step 1: We first generate a set of points in $\mathbb{R}^k$ with coordinate matrix X in such a way that its gram matrix will equal to M. To do this, diagonalize M using an orthogonal matrix Y such that $L = YMY^T$ has the eigen value of M in its first k diagonal entries and zeros elsewhere. Now take $\tilde{X} = L^{\frac{1}{2}}Y$

Step 2: We show that $\tilde{X}$'s gram matrix $G = \tilde{X}^T\tilde{X}$ is in fact M.

$$G = \tilde{X}^T\tilde{X} = X^TX = Y^T(L^{\frac{1}{2}})(L^{\frac{1}{2}})Y = Y^TLY = M$$

So from proof $\Rightarrow$ it will follow that metric matrix of X is in fact M. $\square$

More Discussion:

Proof $\Leftarrow$ gives a linear algebra method for finding realization X from metric matric M. Can our traditional method of realization achieve the same complexity? in terms of the following two aspect:

- Defeat the combinatorial exploration of SRCC caused by 2 possible solutions at each step of solving.
- Has to avoid having nested radicals of square roots

We have showed that Cayley-Menger conditions are necessary for embeddibility in dimension K in 3.1.1, now we show that Cayley-Menger condition(which enforce low rank, symmetric and positive eigen values) are in fact sufficient for embedding.
$\square$

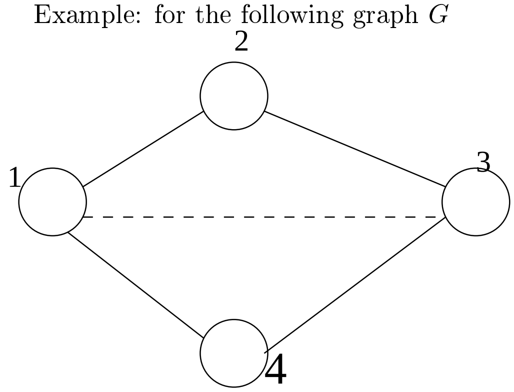
3.2 Question 4

Question 1 and 4 are equivalent in the sense that if we understand one of them, we understand the other.

3.2.1 Observation: Since we only care about euclidian invariants[3] of a realization, not the actual (x, y) coordinates, we can identify completely a realization of (G, d) with sufficiently many of distance $d_{\bar{G}}$, i.e. a set s of distance $\subseteq d_{\bar{G}}$ such that $s \bigcup d_G$ generates the entire ring of euclidian invariants or $|G|$ points.

Def: If polynomial $P_1 \cdots P_n$ generate q, if $\exists polynomials\, q_1 \cdots q_n$ such that $\sum q_i P_i = q$.

Example: for the following graph G



$d_G = \{d_{12}, d_{23}, d_{14}, d_{34}\}$ $\quad d_{\bar{G}} = \{d_{24}, d_{13}\}$
$d_{13} \in [d_{12} - d_{23}, d_{13} + d_{23}] \bigcap [d_{14} - d_{13}, d_{14} + d_{13}]$

Let's identify a realization with a value of d_{13}, while it is not complete, because $d_G \bigcup \{d_{13}\}$dose not generate all Euclidian invariants,i.e. the map from $I_{d_{13}} \to \mathbb{R}^{2\times 4} / euclidian\, group$ is not a real map, however, either d_{13} or d_{24} is representation of realization space that is sufficient to decide emptyness.

3.2.2 Facts: For many applications, they need only sufficiently many $Q \subseteq d_{\bar{G}}$ constraints, so that given $(Q \bigcup d_G)$ forces only finitely many possibilities for $d_{\bar{G}} \setminus Q$ in order for a realization to exist.

Ex. Decide emptyness of realization space

Ex. "Walking" or sampling configuration space, for instance of molecules in molecular dynamics.

If the realization space represented by the distance value for Q is "simple", Eg. a linear polytope, "Walking" is easy.

[3] distance