Geometric Constraint Lecture(Mar 21-23)

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Mar 21, 2006

1 Problem Categories

2 Five Questions

1. Given graph G, characterize d for which (G, d) has a realization. Here d are constraints, for example distance constraints.

2. Given graph G and constraints d, provide a realization.

3. Given grpah G, generically classify it into two categories:

- It has finite number of realizations.
  - One realization
  - Many realizations
- It has infinite number of realizations.

4. Given G, generically characterize the realization space.

5. Given nongeneric G, with fixed or restricted d, answer question 3 and 4. Give the classification and description of its realization space.

3 Working on these Five Questions

3.1 Question 1

Problem: **G is a complete distance graph, find** $\{d : (G, d)\ \textit{has a realization in}\ \mathbb{R}^k \textit{space}\}$.

Theorem: Cayley-Menger conditions are the necessary and sufficient conditions that (G, d) has a realization in $\mathbb{R}^k$space.

3.2 Question 4

Question 1 and 4 are equivalent in the sense that if we understand one of them, we understand the other.

3.3 Question 3

3.3.1 Leman's theorem: A graph G generically has only finitely many solutions iff the following two conditions hold:

1. $\forall subgraph\ S \subseteq G,\ 2|V_S| - |E_S| \geq 3$
2. $2|V_G| - |E_G| = 3$

3.3.1.1 General Leman theorem: A graph $G = (V, E)$ generically has at most finitely many solutions iff $\exists subgraph\ G' = (V, E')\ with\ E' \subseteq E$ such that

1. $\forall subgraph\ S \subseteq G',\ 2|V_S| - |E_S| \geq 3$
2. $2|V_{G'}| - |E_{G'}| = 3$

3.3.1.2 Definition of Generic

Embedding: can be understanded in the following three ways:

- $(x_1 y_1 x_2 y_2 \cdots x_n y_n) \subseteq \mathbb{R}^{2n}$
- $d_{\bar{G}} \subseteq \mathbb{R}^{|V_{\bar{G}}|}$
- $\mathbb{R}^{2n} \diagdown E_2$

Given (G, d) **o/w** d_G
$\{(x_1 y_1 x_2 y_2 \cdots x_n y_n) :\ (x_v - x_w)^2 + (y_v - y_w)^2 = d_{vw}^2\ \forall (v, w) \in E(G) \diagdown E_2\}$
$\leftrightarrow$
$d_{\bar{G}} = \{(x_a - x_b)^2 + (y_a - y_b)^2, \cdots (a, b) \notin E(G),\ :\ (x_v - x_w)^2 + (y_v - y_w)^2 = d_{vw}^2\ \forall (v, w) \in E(G)\}$

There is one to one map between these two sets.

Typical embedding: An embedding Q of G is generic if

$\exists a\ small\ enough\ neighborhood\ of\ (d_{Q,G}),\ d_{Q,G\pm\xi},\ all\ realization\ of\ (G, d_{Q,G\pm\xi})\ are\ rigid \Leftrightarrow Q\ is\ rigid$**.**

Alternately, One can define a small enough neighborhood of Q itself, Require all corresponding realization to be rifid for their corresponding distance values.

3.3.1.3 Rigidity of Graph:

Def1: A Graph is rigid $\exists a\ generic\ embedding\ that\ is\ rigid.$

Def2: A Graph is rigid if $all\ generic\ embeddings\ are\ rigid.$

These two definitions turn out to be equivalent and we will give the proof in the following notes.

Def3: A Graph is globally rigid if $it\ is\ rigid$ &

$\forall generic\ embedding\ Q\ with\ distances\ d_{G,Q},\ Q\ is\ the\ unique\ generic\ realization\ of\ (G, d_{G,Q})$

Def4: An embedding Q of G is rigid if

$\exists\ a\ small\ enough\ \mathbb{R}^{2n} - neighborhood\ Q_\xi,\ such\ that\ for\ \forall Q' \in Q_\xi,\ Q'\ is\ a\ realization\ of\ (G, d_{Q,G}) \Leftrightarrow Q'\ is\ a\ rigid\ motion\ of\ Q.$

Def5: An embedding Q of G is generic if

$\exists\ a\ small\ enough\ \mathbb{R}^{2n} - neighborhood\ Q_\xi,\ such\ that\ \forall Q' \in Q_\xi,\ Q'\ is\ rigid \Leftrightarrow Q\ is\ rigid.$

Question: Given a particular embedding $Q \in \mathbb{R}^{2n}\ of\ G$, Decide

1. is Q generic?
2. is Q rigid?

3.3.2 Jackson-Jordon theorem

for $d = 2$, $G = (V, E)$ generically has an unique solution (globally rigid) $\Leftrightarrow G$ is redundantly rigid & 3-connected or it is a triangle.

Redundantly rigid: removal of any edge preserves rigidity of G.

3.3.2.1 Hendrickson's theorem

$G = (V, E)$ **is globally rigid in** d **dimension** $\Leftrightarrow G$ **is reduntantly rigid for** d **dimension &** $(d+1)$ **connected.**

$\Leftarrow$**proved**

$\Rightarrow$**Conely disproved for** $d \geq 3$

$\Rightarrow$**proved (Jackson-Jordon theorem)**

3.3.3 Owen's theorem

A graph is quadratically solvable $\Leftrightarrow$**it is not 3-connected.**

Quadratically solving: A consraint system (G, d) is quadratically solvable if it is triangularizable into quadratics.

$\Leftarrow$**proved**

$\Rightarrow$**For planar graph, a graph is quadratically solvable** $\Rightarrow$**it is not 3-connected**

$\Rightarrow$**For general graph, open problem**