Final (COT3100, Sitharam, Spring 2017, 2 hours)

NAME:last________________________first:________________________

UF-ID

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(1) In a group of n people a *celebrity* is defined to be a person who knows no one, and is known to everyone. Consider the problem of identifying a celebrity (should one exist) by asking questions of the form: “Excuse me, do you know that person over there?” Show that $\Omega(n)$ questions are neccessary, and $O(n)$ questions are sufficient. (How many celebrities can possibly exist in a group?)

(2)

1. Consider table of non-negative integers with 5 rows and 4 columns such that every row has at least 3 non-zero elements. Show that at least one of the columns must add up to at least 4.

2. In a class of 7 people, there are 14 pairs of people who know each other. Show that there must be 1 person in the class that knows at least 4 people.

3. According to my charted ancestral tree, I have 1 sibling, and 510 cousins. A "cousin" is one with whom I share a grandparent or great-grandparent etc.. Furthermore, each one of my relatives on this tree had (has) exactly one sibling. How many generations does my charted ancestral tree contain? Why?

(3) We call two sets A and B *separable* when there is a set C satisfying: $A \cap C = A$, and $C - A \neq \emptyset$, and $C \cap B = \emptyset$. **Show that A and B are not separable if and only if $A = \overline{B}$ or $A \cap B \neq \emptyset$.** *Hint:* Use a Venn diagram of separable sets to gain intuition. Utilize negation. Note: A diagram alone is not a proof.

(4) Write algorithms (pseudocodes) and prove their correctness, for the following algorithmic problems.

Input: a finite list S of pairs (a, b), where a, b are positive integers.

(i) *Output of Algorithm 1:* A minimum sized set H such that H contains at least one element from each pair in S.

(ii) *Output of Algorithm 2:* A longest sequence of the form $(e_1, e_2), (e_2, e_3), \ldots,$ consisting of pairs in S, given one of the following (write 2 versions of Algorithm 2):

(a) Restriction: e_i are all distinct.

(b) No restriction: e_i may or may not be distinct.

Hint: Use graph theory.

(5) Let $\mathbf{N}$ denote the set of non-negative integers.
(i) Let the subset S_m of $\mathbf{N} \times \mathbf{N}$ be defined as the set of ordered pairs (i, j) such that $i + j \leq m$. Give an expression for the cardinality of S_m in terms of m.
(ii) Let the subset S_m of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consist of all ordered triples (i, j, k) such that $i + j + k \leq m$. Give an expression for the cardinality of S_m in terms of m.
Hint: Draw a diagram of S_4 to develop some intuition. Fix a value of i and count the number of j's for that i. Fix a value of i and j and count the number of k's for that (i, j) and repeat for j. Write a summation formula with 2 or 3 summations. Then use known summation identities to get S_m.

(6) Show that the number of ways of picking e elements out of b boxes, each containing k elements, such that at least 1 element is picked from each box

$$= \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} \binom{jk}{e}.$$

Assume all elements are distinct

Hint: one way to do this is to prove by induction on b, using inclusion-exclusion for the induction step.

(7)

1. What is the probability that there is some set of 5 questions that at least two of you will answer on this test (each of you may have answered more than those 5 questions)? Assume there are 100 of you. Give an expression, you need not evaluate. Explain your expression.

2. A man on the street offers to play this card game with you. In each round, you draw a card from a shuffled, standard deck of 52. Before the next round, the card is put back and the deck thoroughly shuffled. In each round, he will pay you: \$8 if you draw a picture card (Jack, Queen or King); \$11 if you draw an ace; and if you draw an odd numbered card, he will pay you the number on the card, in dollars. However, if you draw an even numbered card, you have to pay him double the number on the card, in dollars.

   (a) Do you expect to gain or lose money if you play the game?

   (b) What is your expected gain (or loss) if you play n rounds? Give an expression in terms of n.

   (c) Give reasons for both above answers.

Bonus 1: Use Question 6 to find the simplest expression you can for:

$$\sum_{x=0}^{k^2}\sum_{y=0}^{k} f(y)\binom{k}{y}\sum_{j=0}^{y}(-1)^{y-j}\binom{y}{j}\binom{jk}{x},$$

where $f(y) = \frac{1}{2^{yk}}(1-\frac{1}{2^k})^{-y}$, and k is any constant positive integer.

Bonus 2:

Consider an undirected graph $G = (V, E)$ with the function $f : V \to N$, such that if $v_1, \ldots, v_k$ is any path in G with distinct vertices, (i.e, $v_i \neq v_j$ for $1 \leq i, j \leq k$), then either $f(v_1) > f(v_2) > \ldots > f(v_k)$, or $f(v_1) < f(v_2) < \ldots < f(v_k)$ holds. Show that such a graph has at most $n - 1$ edges, where $n = |V|$.
Hint: first show that such a graph G cannot have cycles. Then assume that G is connected (this maximizes the number of edges, and is hence the worst case that one has to consider), and prove, by induction on n, that G can have at most $n - 1$ edges.