EXAM 6 (COT3100, Sitharam, Spring 2015)

NAME:last________________________ first:________________________

UF-ID ____

NOTE: You have 60 minutes, please plan your time.
Recall $\binom{n}{r}$ denotes the number of ways of picking a set of r objects from a set of n distinct objects.

(1) Recall the Monty Hall puzzle: as game show participant, you get to pick one out of $n \geq 3$ doors that you guess has the single prize. The game show host opens $k \leq n - 2$ doors which do not have the prize. He then permits you to switch to another door or stay with the door you have selected.

Recall from class that for $n = 3$, $k = 1$, you at least double your probability of winning the prize by switching.

1. Are there n, k for which you should not switch? Why or why not?
2. Given three expressions, in terms of n, k, for (a) the probability that you win after staying and (b) the probability that you win after switching (c) the probability that you lose. *Note: the above probabilities must add up to 1*

(2) You are given a large unlimited supply of identical beads in r distinct colors. Given a color, if you pick a bead out of the supply while blindfolded, the probability that you pick a bead of that color is *always* $1/r$. You construct a necklace of n beads while blindfolded, picking out beads one by one. Bead necklaces have distinct left and right ends - they have a loop on the left and a hook on the right. Give all answers below in terms of n and r.

1. What is the probability that your bead necklace has the same number of beads of each color?

2. Your friend has already constructed a necklace of n beads. What is the probability that the necklace of n beads that you construct while blindfolded is identical to that of your friend's? Why? *Hint:* what is the number of distinct necklaces you can construct?

3. You and your friend construct separate necklaces of n beads while blindfolded. What is the probability that your necklaces turn out to be identical? Why?

4. What is the expected number of beads in your necklace that are neither red nor blue? Why?

(3) A man on the street offers to play this card game with you. In each round, you draw a card from a shuffled, standard deck of 52. Before the next round, the card is put back and the deck thoroughly shuffled. In each round, he will pay you: \$8 if you draw a picture card (Jack, Queen or King); \$11 if you draw an ace; and if you draw an odd numbered card, he will pay you the number on the card, in dollars. However, if you draw an even numbered card, you have to pay him double the number on the card, in dollars.

1. Do you expect to gain or lose money if you play the game?
2. What is your expected gain (or loss) if you play n rounds? Give an expression in terms of n.
3. Why?

(4) The probability of Mr. Smart attending class is p, and not attending class is $1-p$. If he does not attend class, the probability that he does well on the test is e_n. If he attends class, the probability that he does not do well on the test is e_a.

Give an expression, in terms of p, e_n and e_a, for the probability that he attended class, given that he did well on the test. *Hint:* First give an expression for the probability that he did well on the test.

Bonus 1: Let $X(s)$ be a random variable taking non-negative integer values for all events s. Show that $E(X) = \sum\limits_{k=1}^{\infty} \text{Prob}[X(s) \geq k]$ Here we are taking the probability of the event that $X(s) \geq k$. *Hint:* Start with the definition $E(X) = \sum\limits_{k=1}^{\infty} k \ \text{Prob}[X(s) = k]$ and rearrange terms.

Bonus 2: We say k random variables $X_1, \ldots, X_k$ are mutually independent if $\text{Prob}[\bigwedge_{i=1}^{k} (X_i = r_i)] = \prod_{i=1}^{k} \text{Prob}[X_i = r_i]$ Show that $E(\prod_{i=1}^{k} X_i) = \prod_{i=1}^{k} E(X_i)$. *Hint:* First prove it for $k = 2$.