EXAM 4 (COT3100, Sitharam, Spring 2017)

NAME:last____________________________first:____________________________

UF-ID ____

NOTE: You have 2 hours, please plan your time.
Recall $\binom{n}{r}$ denotes the number of ways of picking a set of r objects from a set of n distinct objects. For all integers n, $\binom{n}{n} = \binom{n}{0} = 1$; $0! = 1$; When $r > n$, $\binom{n}{r} = 0$.

(1)

1. What is the minimum number of people that should be put in a room to guarantee that 2 people in the room have the same birthday (birth month and date, e.g: January 1).
2. What is the minimum number of people that should be in a room to guarantee that at least k people were born on the same day of the week (e.g. Wednesday)?
3. What is the minimum number of people that should be put in a room to guarantee that the birth date (e.g, 31st - month and year left out) of at least k people have the same remainder when divided by r. Express your answer in terms of k and r. *Hint:* Consider two possible intervals for the value of r.

Solution.

1. is a straightforward application of Pigeon Hole. 2 and 3 come from the generalized Pigeon Hole.
(1) 367, noticing that Feb 29 is a vaild b'day
(2) $7(k-1)+1$
(3)If $r \geq 31$, $31(k-1)+1$. If $r < 31$, $r(k-1)+1$

(2) Prove that

1. $\sum_{j\ even} \binom{n}{j} = \sum_{j\ odd} \binom{n}{j}$
$= 2^{n-1}$;
2. for all m, n $\binom{n+m}{r} = \sum_{j} \binom{n}{j}\binom{m}{r-j}$.

Solution 1. This is worked out in the book – use binomial theorem It is sufficient to show that the LHS minus RHS = 0 and that LHS+RHS = 2^n. I.e, that $\sum(-1)^j\binom{n}{j} = 0$ and that $\sum\binom{n}{j} = 2^n$ But by the binomial theorem, the first is just $(1-1)^n = 0$ and the second is $(1+1)^n = 2^n$.

2. This is also worked out in the book. The LHS is the number of ways of choosing r things out of $n+m$ things (also called distributing $n+m$ distinct objects into r identical boxes with the 1-1 constraint, so only r of the objects will fit). This can be done by splitting the process into two parts: first pick j objects from the first n objects followed by picking $r-j$ objects from the m objects. Then try all possible j's. This covers all possibilities and every possibility is covered only ones. So the set of possibilities for a fixed j is a crossproduct of the two parts, so the product rule can be applied. Then the sum rule can be applied to sum over all the j's. Notice now that this is exactly the RHS.

(3) Let $S(n,k)$ be the number of ways of partitioning the set $N := \{1,\ldots,n\}$ into a collection of exactly k nonempty disjoint subsets (whose union is N).

1. Show that $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

2. Give a recursive algorithm for listing all the possible partitions.

Solution. 1. Count the partitions of N into k non-empty subsets according to whether x_n (last element) is in a subset by itself. If so, then there are $S(n-1,k-1)$ ways to partition the remaining elements of N. If not, then there are $S(n-1,k)$ ways to partition the remaining elements of N in to k subsets and then k ways to choose the subset to place x_n in to.

(ii) The above gives the algorithm, more or less. For a more detailed pseudocode:

```
function list(n, k, f) {
.       if(n == 0&&k == 0){f([ ]); return;}
.       if(n == 0||k == 0) { return; }
.       list(n − 1, k − 1, function(s) {
.          f(s.concat([[n]])); // append [n] to the array
.       });
.       list(n − 1, k, function(s) {
.          for (i = 0; i < k; i++) {
.             f(s.map(function(t, j){ // append n to the i'th subarray
.                return t.concat(i == j?n :[ ]);
.             }));
.          }
.       });
}
```

define a function to print the list

function print(s) { print(s); }

get the output by calling the following function list(n, k, print);

(4) You are given a large unlimited supply of identical beads in r distinct colors. Given a color, if you pick a bead out of the supply while blindfolded, the probability that you pick a bead of that color is *always* $1/r$. You construct a necklace of n beads while blindfolded, picking out beads one by one. Bead necklaces have distinct left and right ends - they have a loop on the left and a hook on the right. Give all answers below in terms of n and r, with different cases if necessary.

1. What is the probability that your bead necklace has the same number of beads of each color? *Hint:* what is the number of distinct necklaces you can construct? First consider the case of just 2 colors.

2. Your friend has already constructed a necklace of n beads. What is the probability that the necklace of n beads that you construct while blindfolded is identical to that of your friend's? Why?

Solution. 1. The probability is the number of ways to pick a necklace with the same number of beads of each color,

divided by the total number of distinct necklaces.

Assume $k = n/r$ is a positive integer. Otherwise, the probability is zero.

In general, the numerator is computed using the number of ways to distribute identical objects of r different types and n distinct boxes, with k objects of each type (similar to the MISSISSIPI problem).

To pick an n-bead necklace with the same number of beads of each color,

we first choose k positions among the n positions on the necklace for the first color ($\binom{n}{k}$ ways),

then choose k positions from the remaining positions for the second color ($\binom{n-k}{k}$ ways),

etc.

The total number of ways is

$$\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\cdots\binom{2k}{k}\binom{k}{k} = \frac{n!}{(k!)^r}$$

The denominator is the different ways to distribute n distinct objects (the positions on the necklace) into the r distinct boxes (colors) without the 1-1 condition. Since each position on the necklace has r different choices of colors, the total number of distinct necklaces is r^n.

Therefore the overall probability is $\frac{n!}{(k!)^r r^n}$.

2. Since the total number of distinct necklaces is r^n, the probability of picking a specific one — the one identical to your friend's — is $\frac{1}{r^n}$.

(5) Recall the Monty Hall puzzle: as game show participant, you get to pick one out of $n \geq 3$ doors that you guess has the single prize. The game show host opens $k \leq n-2$ doors which do not have the prize. He then permits you to switch to another door or stay with the door you have selected.

Recall from class that for $n = 3$, $k = 1$, you at least double your probability of winning the prize by switching.

1. Are there n, k for which you should not switch? Why or why not?

2. Give three expressions, in terms of n, k, for

   (a) the probability that you win after staying and

   (b) the probability that you win after switching

   (c) the probability that you lose.

   Note: the above probabilities must add up to 1

Solution

1. There are not, we can see this by comparing (a) and (b) below and seeing that $n - 1 \geq n - k - 1$ when $k \geq 0$. In general, if $k = 0$, then the probability for winning in each case is the same.

2. (a) $\frac{1}{n}$

   (b) The probability of winning after switching is the probability that you do not pick the door the first time times the probability that you pick the door after the switch. After the switch, there are $n - k - 1$ doors to choose from:
   $\left(\frac{n-1}{n}\right)\left(\frac{1}{n-k-1}\right)$

   (c) From the hint, we can deduce the probability of losing is $1 - \frac{1}{n} - \left(\frac{n-1}{n}\right)\left(\frac{1}{n-k-1}\right)$

(6) The probability of Mr. Smart attending class is p, and not attending class is $1 - p$. If he does not attend class, the probability that he does well on the test is e_n. If he attends class, the probability that he does not do well on the test is e_a.

Give an expression, in terms of p, e_n and e_a, for the probability that he attended class, given that he did well on the test. *Hint:* First give an expression for the probability that he did well on the test.

Solution Probability that he did well on the test $= p(1 - e_a) + (1 - p)e_n$

By applying Bayes' theorem

$P(attended \mid didwell) = p(1 - e_a)/(p(1 - e_a) + (1 - p)e_n)$

Bonus 1: In the bead problem above, you and your friend construct separate necklaces of n beads while blindfolded. What is the probability that your necklaces turn out to be identical? Why?

Solution

The probability is still $\frac{1}{r^n}$.

Given a specific necklace N, the probability that both you and your friend construct N is $\frac{1}{(r^n)^2}$.

But since there are r^n different choices for N, by the sum/union rule, we need to multiply $\frac{1}{(r^n)^2}$ by r^n.

Bonus 2: Prove that $\binom{n+m}{m} = \sum_j \binom{m+1}{j}\binom{n-1}{j-1}$

Solution Just an application of Problem 2. Notice that proof holds even if instead of splitting $n+m$ objects into n and m, we instead split into $m+1$ and $n-1$. Also notice that on the LHS $\binom{n+m}{m} = \binom{n+m}{n}$ and that on the RHS, $\binom{n-1}{n-k} = \binom{n-1}{k-1}$. This completes the proof.

Another purely combinatorial way Think of the LHS as ways of distributing n identical objects into $m+1$ distinct boxes. The RHS is writing the LHS as the union of sets of ways where exactly j of the boxes are nonempty. Then $\binom{m+1}{j}$ is the number of ways to pick the nonempty boxes in which to put one object each (so they are nonempty). This is multiplied (by the product rule) with the number of ways of distributing the remaining $n-j$ identical objects into j distict boxes which is $\binom{n-j+j-1}{j-1} = \binom{n-1}{j-1}$ Now take the union (and hence sum) over all j.