EXAM 3 (COT3100, Sitharam, Spring 2017)

NAME:last________________________first:________________________

UF-ID ____

Section ____

NOTE: You are not required to use induction. However, using induction will help obtain systematic partial credit. Write down (1) what variable your induction is on (2) induction basis (3) induction hypothesis (4) statement of induction step (5) proof of induction step. All carry partial credit. You have 2 hours, please plan your time. Problems are not ordered by difficulty.

(1)

1. Show that any $2^n \times 2^n$ checkerboard with any 1×1 square removed can be exactly covered without overlaps by right triominoes (a right triomino is an L-shaped piece with three 1×1 squares).

2. Give and prove a recursive formula for the number of triominoes required.

Feel free to use pictures to illustrate your writing, but the key points of the proof, especially of the induction step, have to be written very carefully. *Hint:* This problem was done in class.

Solution This problem has been solved in the book. And in the Rosen-provided lecture power point slides available to students.

(2) Show that $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$. *Hint:* Use (or first derive) an explicit formula for the RHS.

Solution

http://math.stackexchange.com/questions/996083/induction-proof-for-a-summation-sum-i-1n-i3-
996100

(3) Show by induction on n that the set of elements that belong to an odd number of sets $A_1 \dots A_n$ is exactly $A_1 \oplus A_2 \oplus \dots A_n$, where $A \oplus B := (A \cap \overline{B}) \cup (\overline{A} \cap B)$. *Hint*: A proof of this problem was sketched in class. To develop intuition, start with $n = 2$. Work out how the statement for $n = 3$ follows from the statement for $n = 2$. Notice that $A_1 \oplus \dots \oplus A_n = (A_1 \oplus \dots A_{n-1}) \oplus A_n$.

Solution. Basis step : when n=1, $\bigoplus_{i=1}^{1} A_i$ is true if $x \in A_1$ or is false other wise.

IH : set of elements that belongs to an odd number of sets $A_1 ... A_n$ is given by $\bigoplus_{i=1}^{n} A_i$, where $A \bigoplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$

$$\bigoplus_{i=1}^{n+1} A_i = (\bigoplus_{i=1}^{n} A_i \cap \bar{A}_{n+1}) \cup (\overline{\bigoplus_{i=1}^{n}} A_i \cap A_{n+1})$$

$(\bigoplus_{i=1}^{n} A_i \cap \bar{A}_{n+1}) \cup (\overline{\bigoplus_{i=1}^{n}} A_i \cap A_{n+1})$ is true if an element belongs to odd number of sets from $A_1 ... A_n$ and not belongs to A_{n+1} or belongs to even number of sets from $A_1 ... A_n$ and belongs to A_{n+1}. Both cases means element is belongs to odd number of sets from $A_1 ... A_{n+1}$.

$(\bigoplus_{i=1}^{n} A_i \cap \bar{A}_{n+1}) \cup (\overline{\bigoplus_{i=1}^{n}} A_i \cap A_{n+1})$ is false if an element belongs to even number of sets from $A_1 ... A_n$ and not belongs to A_{n+1} or belongs to odd number of sets from $A_1 ... A_n$ and belongs to A_{n+1}. Both cases means element is belongs to even number of sets from $A_1 ... A_{n+1}$.

(4) Show that the number of factors of n with prime decomposition $n = 2^{c_2} 3^{c_3} 5^{c_5} \dots$ is the product $(c_2 + 1)(c_3 + 1)(c_5 + 1)\dots$. *Hint:* A noninductive proof of this problem was given in class. Use strong induction with inductive hypothesis for a number $m < n$ whose prime decomposition is identical to that of n except that it omits the largest prime appearing in n.

Solution The hint tells you what to do. The base case is straight forward. Take $n = 2$. For the induction hypothesis, let $m = 2^{c_2} 3^{c_3} 5^{c_5} \dots p_{k-1}^{c_{k-1}}$, where p_{k-1} is the penultimate prime appearing in n, p_k is the largest prime appearing in n, with power c_n and $m = n/p_k^{c_k}$. By strong induction, the statement holds for m. Now for the induction step: what factors does n have? For each factor of m, a factor of n can be created by multiplying with $p_k^0, p_k^1 \dots p_k^{c_n}$. This is exactly the number of factors of m times $c_k + 1$, thereby proving the induction step.

(5) Show the divisibility rule: "a number m is divisible by 2^n if and only if the number formed by the last n digits of m, i.e, $m \bmod 10^n$, is divisible by 2^n."

Solution Without induction is easier: The number $m = m_1 * 10^n + m \bmod 10^n$ is $0 \bmod 2^n$ if nd only if $m \bmod 10^n$ is $0 \bmod 2^n$ since $m_1 * 10^n$ is $0 \bmod 2^n$.

(6) Show that n lines partition the plane into $n(n + 1)/2 + 1$ regions if no three lines meet at common point and no two lines are parallel. *Hint:* A proof of this problem was sketched in class. Build intuition with small values of n. Consider how many new regions the n^{th} line creates as it intersects the previous $n - 1$ lines.

Solution Prove by induction.

Basis step: when $n = 1$, the plane is partitioned into $n(n + 1)/2 + 1 = 2$ regions, thus the statement holds.

Induction hypothesis: assume for some positive integer k, k lines partition the plane into $k(k + 1)/2 + 1$ regions.

Inductive step: consider adding a $(k+1)$th line to the plane. Since no two lines are parallel, the $(k + 1)$th line intersects with all the previous k lines, creating k new intersection points as no three lines meet at common point.

The k new intersection points give $k+1$ segments on the new line, that is, the new line cuts through $k+1$ original regions, creating $k+1$ new regions on the plane. So $k+1$ lines partition the plane into $k(k+1)/2+1+(k+1) = (k+1)((k+1)+1)/2+1$ regions, thus the statement holds for $k+1$ lines.

Bonus 1: Use Problem 4 to show that n planes partition 3 dimensional space into $(n^3+5n+6)/6$ regions, if no more than 3 planes intersect at any point, and every 3 planes intersect at a point. *Hint:* Consider the partition of the n^{th} plane by the lines of intersection of the n^{th} plane with the previous $n-1$ planes.

Solution.

Proof by induction.

Basis step: when $n=1$, the 3d space is partitioned into$(n^3+5n+6)/6 = 2$ regions, thus the statement holds.

Induction hypothesis: assume for some positive integer $i-1$, $i-1$ planes partition the 3d space into $((i-1)^3+5(i-1)+6)/6$ regions.

Inductive step: Consider adding the i^{th} plane and counting how many new regions it creates. Two non parallel planes intersect along a line, so the other $i1$ planes all produce intersection lines on i^{th} plane. The conditions that no two planes are parallel, any three have one point in common, and no four have a common point imply that on i^{th} plane, no intersection lines are parallel and no three lines have a common point. Thus we can apply the two-dimensional result and say that plane i is divided into $i(i-1)/2+1$ regions by the i 1 lines produced from the intersections of the other planes. Each region of i^{th} divides one of the existing regions of 3-space created by the first i 1 planes in two, so adding the i^{th} plane adds $i(i-1)/2+1$ regions.

so the new total regions $= ((i-1)^3+5(i-1)+6)/6 + i(i-1)/2+1 = (i^3+5i+6)/6$

Bonus 2: Give and prove an expression for the sum of factors of n with prime decomposition $n = 2^{c_2}3^{c_3}5^{c_5}\ldots$. Your expression can use p_i's and c_i's, where p_i is the i^{th} prime and c_i is the power with which it appears in n. *Hint:* A noninductive proof of this problem was given in class. Use strong induction with inductive hypothesis for a number $m < n$ whose prime decomposition is identical to that of n except that it omits the largest prime appearing in n.

Solution The hint tells you what to do. A factor is a product of arbitrary powers of p_i (between 0 and c_i) for each prime p_i that appears in n. Sum of these products can be expressed as a product of sums, so $\prod_i^k (1+p_i^1 \ldots p_i^{c_i})$, where p_k is the largest prime appearing in n - is a good conjecture for the expression for sum of factors of n. The base case is straight forward. Take $n=2$. Let $m = 2^{c_2}3^{c_3}5^{c_5}\ldots p_{k-1}^{c_{k-1}}$, where p_{k-1} is the penultimate prime appearing in n, p_k is the largest prime appearing in n, with power c_k and $m = n/p_k^{c_k}$. By strong induction, the statement holds for m. What factors does n have? For each factor of m, a factor of n can be created by multiplying with each of $p_k^0 = 1, p_k, \ldots, p_k^{c_k}$. So the new sum of factors is the sum of factors of m times $(p_k^0 + p_k^1 \ldots + p_k^{c_k})$. This proves the induction step.