ON STABILITY OF m-VARIATE C^1 INTERPOLATION

by

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1.Introduction

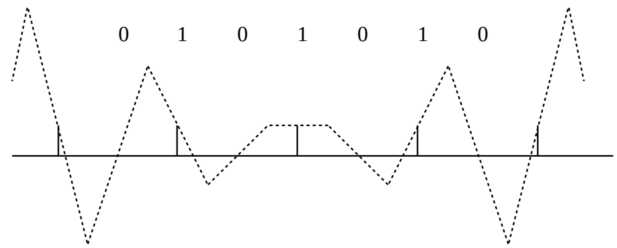
It is well-known that univariate piecewise quadratics do not stably interpolate at breakpoints (see e.g. [B78 Ch. 6], cf. Figure 1.1). Recently, [PS '90] established that bivariate cubics do not stably interpolate values at the vertices of triangulations. Both results are special cases of the result proved below, namely that the process of interpolating at the vertices of an m-dimensional, simplicial complex by a C^1 piecewise polynomial of degree $m+1$ is, in general, unstable if we allow only one polynomial piece per simplex.

To establish the result, it suffices to exhibit, for each m, a sequence of domain decompositions and vertex values such that no corresponding sequence of interpolants is uniformly bounded. In fact, as shown in Section 2, it suffices to exhibit a particular symmetric, shift-invariant decomposition Δ of $\mathbb{R}^m$ with symmetric, shift-invariant values at the vertices to which no bounded C^1 interpolant exists. We give such a decomposition with vertices in $\mathbb{Z}^m$ and show, in Section 3, that a bounded interpolant exists if and only if the alternating sum of the values at the vertices of any unit-cube in Δ is zero:

$$\sum_{\alpha\in\{0,1\}^m} (-)^\alpha f(\alpha) = 0, \qquad \text{where } (-)^\alpha := (-1)^{\alpha_1}\cdots(-1)^{\alpha_m}. \tag{A}$$

A standard diagonalization argument then shows that any sequence of interpolants to a growing, finite subset of the data can be uniformly bounded if and only if (A) holds. Since the data can be viewed as a perturbation of more general data, the instability result follows.

Notation. In the following, k repeated entries are abbreviated as a^k. An *interpolant* is a C^1 piecewise polynomial bounded interpolant with one polynomial piece of degree $m+1$ per simplex. The simplex with the vertices $0, e_1, e_1+e_2, \ldots, \sum_{i=1}^m e_i$, where e_i is the ith unit vector is called σ. The Bernstein-Bézier (short BB) form is used extensively (see e.g. [B87]). For example, the instability of interpolation by univariate C^1 quadratics can be concluded from the unboundedness of the BB-net for 0-1-data (Figure 1.1).



(1.1) Figure: BB-net and instability of univariate C^1 quadratics.

2. A decomposition leading to a shift-invariant, symmetric interpolant

This section explains how shift-invariance and symmetry of a simplicial decomposition Δ of $\mathbb{R}^m$ and of the values at its vertices can be used to generate a shift-invariant and symmetric interpolant.

The simplicial decomposition Δ: Subdivide the m-dimensional unit-cube with vertices in $\{0,1\}^m$ into $m!$ simplices such that the vertices of each simplex are lexicographically increasing. Fill out a 2-cube with 2^m copies of the subdivided unit-cube obtained by rotation about the origin. Shift the 2-cube over the grid $2\mathbb{Z}^m$ to cover $\mathbb{R}^m$.

For example, for $m = 2$, the simplices of the unit-cube are the triangles with vertices

$$\begin{array}{ccc} (0,0) & (1,0) & (1,1) \\ (0,0) & (0,1) & (1,1). \end{array}$$

For $m = 3$, the simplices are the tetrahedra

$$\begin{array}{cccc} (0,0,0) & (1,0,0) & (1,1,0) & (1,1,1) \\ (0,0,0) & (1,0,0) & (1,0,1) & (1,1,1) \\ (0,0,0) & (0,1,0) & (1,1,0) & (1,1,1) \\ (0,0,0) & (0,1,0) & (0,1,1) & (1,1,1) \\ (0,0,0) & (0,0,1) & (1,0,1) & (1,1,1) \\ (0,0,0) & (0,0,1) & (0,1,1) & (1,1,1). \end{array}$$

We note that the set of all vertices of Δ equals $\mathbb{Z}^m$ and that Δ is both invariant with respect to shifts on the $2\mathbb{Z}^m$-grid and symmetric with respect to the origin, hence with respect to the $2\mathbb{Z}^m$-grid. We say, Δ is *symmetric* and *2-shift-invariant.* The decomposition is chosen to make induction easy.

The values V at the vertices of Δ: The vertex values, too, are symmetric and 2-shift-invariant. That is, if $\gamma \in \mathbb{Z}^m$ is a vertex of Δ with value $V(\gamma)$, then

$$V(\gamma) = V(\gamma + 2e_i) \qquad \text{(2-shift-invariance)} \qquad\qquad \text{and}$$

$$V(\gamma_1,\ldots,\gamma_i,\gamma_{i+1},\ldots,\gamma_m) = V(\gamma_1,\ldots,\gamma_{i+1},\gamma_i,\ldots,\gamma_m) \qquad \text{(symmetry)}$$
$$V(\gamma_1,\ldots,\gamma_i,\ldots,\gamma_m) = V(\gamma_1,\ldots,-\gamma_i,\ldots,\gamma_m).$$

The following lemma shows that if an interpolant exists, then a symmetric and 2-shift-invariant interpolant f exists. Such an interpolant is useful, since it can be analyzed by looking at just one simplex, σ, of Δ.

(2.1) Lemma. *If there exists an interpolant to the values V at the vertices of Δ, then there exists a 2-shift-invariant, symmetric interpolant f with the properties*

for $x \in \mathbb{R}^m$, $i \in \{1,\ldots,m-1\}$,

$$f(x_1,x_2,\ldots,x_m) = f(2-x_1),x_2,\ldots,x_m) \tag{d1}$$
$$f(x_1,\ldots,x_i,x_{i+1},\ldots,x_m) = f(x_1,\ldots,x_{i+1},x_i,\ldots,x_m), \quad \text{and} \tag{d2}$$
$$f(x_1,\ldots,x_{m-1},x_m) = f(x_1,\ldots,x_{m-1},-x_m). \tag{d3}$$

Proof. Assume that the BB coefficients of the given interpolant g are bounded by some constant K. Then one constructs a symmetric, 2-shift-invariant interpolant f with bounded coefficients as follows. For $x \in \mathbb{R}^m$ and $\gamma \in 2\mathbb{Z}^m$ define the shift: $E^\gamma g(x) := g(x_1-\gamma_1,\ldots,x_m-\gamma_m)$. Since Δ and V are 2-shift-invariant, $E^\gamma g$ interpolates the data and so does the average of $(n+1)^m$ shifts of g over the m-dimensional $2n$-cube:

$$g_n := \sum_{0\le\gamma_i\le 2n} E^\gamma g/(n+1)^m. \tag{2.2}$$

For $\delta_i \in \{-2,0,2\}$, g_n and $E^\delta g_n$ differ in at most $2((n+1)^m - n^m)$ summands $E^\gamma g/(n+1)^m$. Hence the coefficients of g_n and $E^\delta g_n$ differ by at most $2((n+1)^m - n^m)K/(n+1)^m = O(1/n)$ and therefore, for any $\epsilon > 0$, there exists an n such that

$$\|g_n - E^\delta g_n\| < \epsilon,$$

where the norm measures the maximal BB-coefficient. In other words, the interpolants g_n are asymptotically 2-shift-invariant. A standard diagonalization argument then yields a 2-shift-invariant interpolant h.

To obtain symmetry, we define the rotations

$$R^{ij}h(x_1,\ldots,x_i,x_{i+1},\ldots,x_m) := h(x_1,\ldots,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^j \begin{pmatrix}x_i\\ x_{i+1}\end{pmatrix},\ldots,x_m),$$

compute

$$h_m := \sum_{i=1}^{m-1} \sum_{j=1}^{4} R^{ij} h/4(m-1).$$

and finally average the reflections of h_m to obtain

$$f(x_1, \ldots, x_m) := (h_m(x_1, \ldots, x_m) + h_m(-x_1, \ldots, x_m)$$
$$+h_m(x_1, \ldots, -x_m) + h_m(-x_1, \ldots, -x_m))/4.$$

One checks that f satisfies $(d2)$ by symmetry and $(d1)$ and $(d3)$ by shift-invariance and symmetry. Finally, shifting, averaging and reflection neither increase the degree nor decrease the smoothness of the interpolant. (Averaging can, however, increase the smoothness or decrease the degree.) ♣

We note that the above proof works even if K is not a constant, but a function of n that grows strictly slower than n. Hence, if (A) does not hold, then even a non-uniform, slowly-growing bound on a sequence of interpolants cannot exist.

3. A constraint on the vertex values

This section establishes that the C^1 interpolant f of Lemma 2.1 exists if and only if condition (A) holds. This follows from an analysis of the C^1 constraints: the system of constraints is overdetermined.

The list of $\mathbf{C^1}$ constraints in BB-form: Due to symmetry and shift-invariance of the interpolant f of Lemma 2.1, it is sufficient to consider the C^1 constraints across the facets of σ. By assumption, f restricted to σ is a polynomial of degree $m+1$. This polynomial will be represented in BB-form with coefficients $C(x)$, where $x \in \mathbb{N}^{m+1}$ and $\sum x_i = m+1$, i.e. x is a scaled barycentric coordinate vector. The polynomial $f_{|\sigma}$ has $\binom{2m+1}{m+1}$ coefficients. It is worth noting that f is a function on $\mathbb{R}^m$ whereas C is a function on $\mathbb{N}^{m+1}$.

Smoothness implies $\binom{(m-1)+m}{m}$ constraints for each of the $m+1$ (hyper-)facets of σ. We juxtapose the constraints corresponding to $(d1)$, $(d2)$, and $(d3)$ and their BB representation (for $(D2)$ the $i-1$st, ith, and $i+1$st slot are shown):

$$D_1 f_{|x_1=1} = 0, \qquad C(0,r,\ldots) = C(1,r-1,\ldots) \tag{D1}$$

$$(D_i - D_{i+1}) f_{|x_i=x_{i+1}} = 0, \qquad C(\ldots,r,0,s,\ldots) \tag{D2}$$

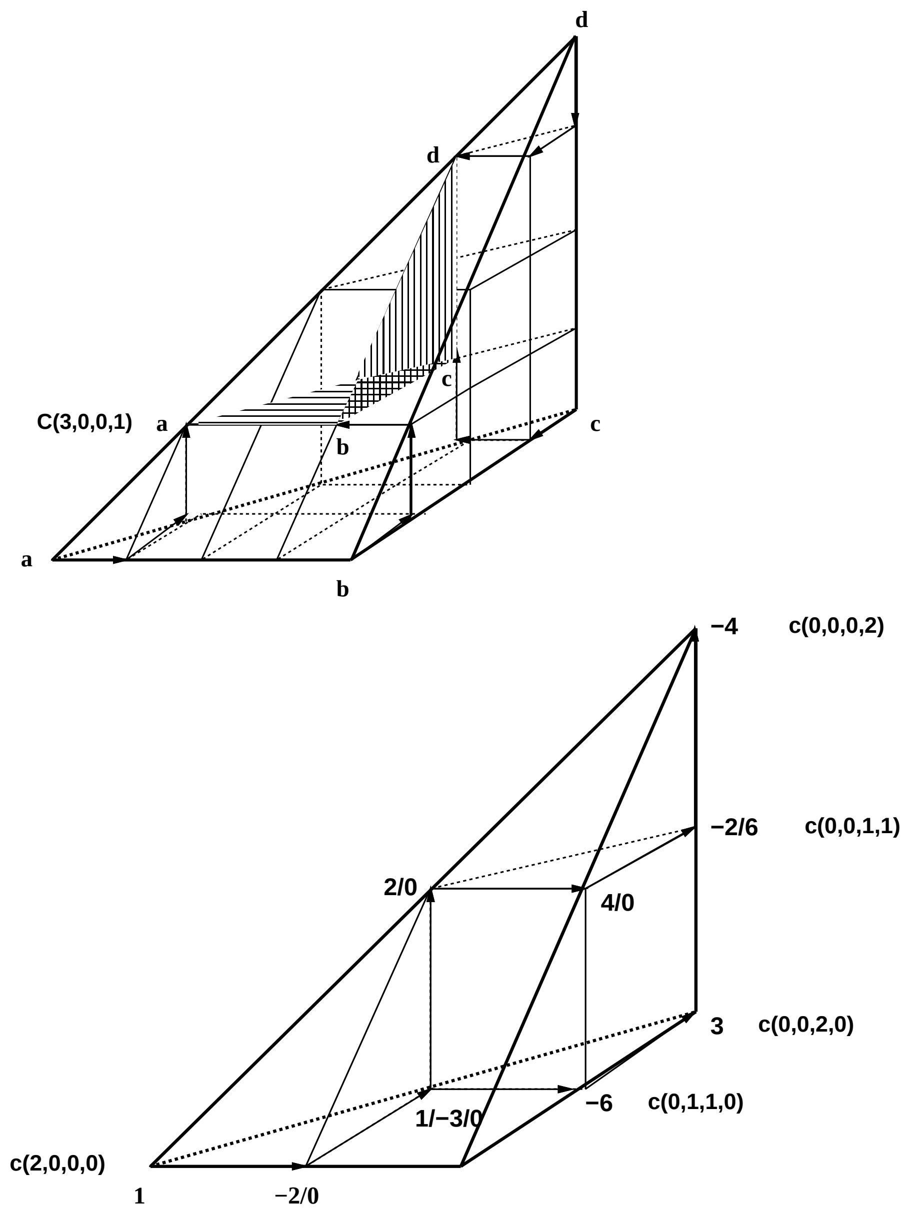
$$= 2C(\ldots,r-1,1,s,\ldots) - C(\ldots,r-1,0,s+1,\ldots)$$

$$D_m f_{|x_m=0} = 0 \qquad C(\ldots,r,0) = C(\ldots,r-1,1) \tag{D3}$$

$$\text{for } r > 0, s \geq 0, i = 1,\ldots,m-1.$$

One checks that the support of each constraint in BB-form is at most 2-dimensional (rather than m-dimensional) since the four vertices $(1^i,0,0,0^{m-2-i})$, $(1^i,1,0,0^{m-2-i})$, $(1^i,0,1,0^{m-2-i})$, $(1^i,1,1,0^{m-2-i})$ for $0 < i < m$ are linearly dependent and each of $(0,0^{m-1})$, $(1,0^{m-1})$, $(-1,0^{m-1})$ and $(1^{m-1},1)$, $(1^{m-1},0)$, $(1^{m-1},-1)$ are linearly dependent. Since $(1^i,1,0,0^{m-i-2})$ and $(1^i,0,1,0^{m-i-2})$ have the same distance from the vertices $(1^i,r,r,0,0^{m-i-2})$, $r \in \{0,1\}$, the weights in $(D2)$ are 2 and -1.

The system of interpolation and smoothness requirements has more variables than constraints. Nevertheless, Theorem 3.11 will show that it is overdetermined. The proof uses the smoothness constraints across the facets to express the vertex coefficients of f in terms of coefficients of f belonging to an $m-1$-dimensional simplex. To motivate the following three preparatory lemmas, we look at the trivariate case in detail.



(3.2) Figure: (top) Reduction of the domain simplex.
(bottom) Expressing $c(2,0,0,0)$ in terms of $c(0,\ldots)$.

(3.1) Example.

For $m = 3$, $f_{|\sigma}$ has 35 coefficients and 30 constraints. The single interior coefficient has the barycentric coordinates $(1,1,1,1)$ and the Cartesian coordinates $(3,2,1)/4$. By

$(D2)$, $(D2)$ and $(D3)$

$$\begin{aligned} C(4,0,0,0) &= 2C(3,1,0,0) - C(3,0,1,0) = 4C(3,0,1,0) - 2C(3,0,0,1) - C(3,0,1,0) \\ &= C(3,0,0,1) =: a \end{aligned}$$

and similarly (cf. Figure 3.2(top))

$$C(0,4,0,0) = C(1,2,0,1), \quad C(0,0,4,0) = C(1,0,2,1), \quad C(0,0,0,4) = C(1,0,0,3).$$

To obtain a constraint on the value of the coefficients corresponding to the vertices of σ, it will therefore suffice to prove a constraint for a quadratic polynomial f^* on a reduced simplex σ^* with vertex coefficients

$$c(2,0,0,0) := C(3,0,0,1), \qquad c(0,2,0,0) := C(1,2,0,1),$$

$$c(0,0,2,0) := C(1,0,2,1), \qquad c(0,0,0,2) := C(1,0,0,3).$$

Within that reduced simplex, repeated application of the constraints of type $(D2)$ yields (cf. Figure 3.2(bottom))

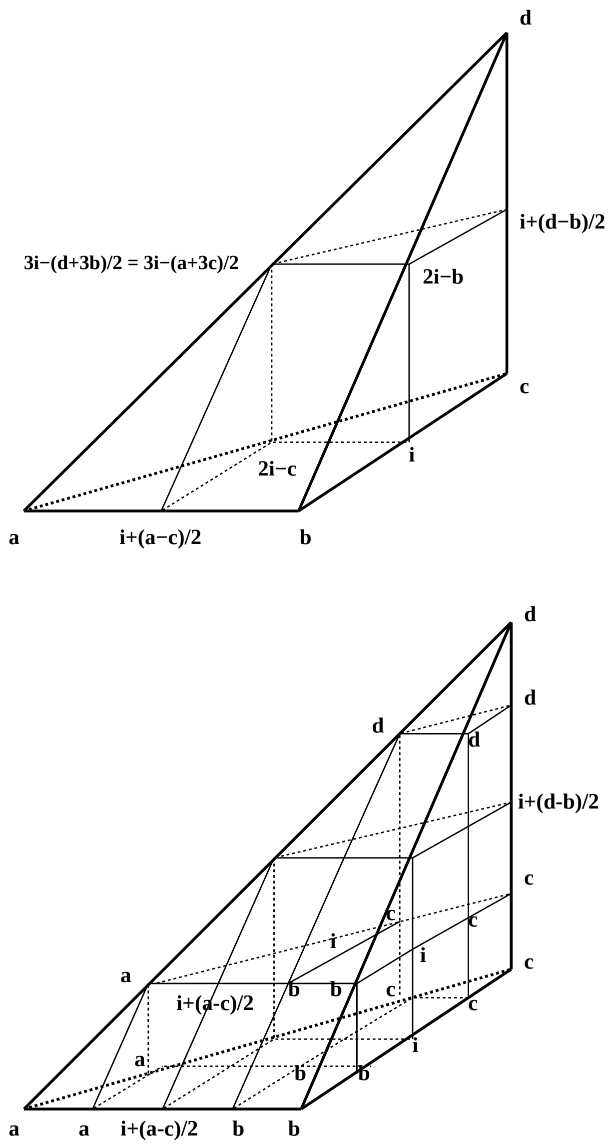
$$\begin{aligned} f(0,0,0) &= c(2,0,0,0) = \\ &= 4c(0,0,0,2) - 3c(0,0,2,0) - 6c(0,0,1,1) + 6c(0,1,1,0) \\ -3f(1,0,0) &= -3c(0,2,0,0) = -3c(0,0,0,2) - 6c(0,0,1,1) - 6c(0,1,1,0) \\ 3f(1,1,0) &= 3c(0,0,2,0) \\ -f(1,1,1) &= -c(0,0,0,2). \end{aligned} \tag{3.3}$$

That is,

$$\sum_{l=0}^{3} (-)^l \binom{m}{l} f(1^l, 0^{m-l}) = f(0,0,0) - 3f(1,0,0) + 3f(1,1,0) - f(1,1,1) = 0$$

must hold, if trivariate C^1 quartics are to interpolate data at the vertices. By symmetry,

$$3f(1,0,0) = f(1,0,0) + f(0,1,0) + f(0,0,1),$$



(3.4) Figure: (top) A solution for the quadratic on the reduced simplex.
(bottom) Extension to a cubic solution to the full simplex.

i.e. the alternating sum of the values at the vertices of the unit-cube must be zero.

Conversely, all the C^1 constraints that involve the vertex coefficients of f^* can be enforced; for $m = 3$, this means that all constraints on f^* are enforced (cf. Figure 3.4(top)). To obtain a solution f, it suffices to extend f^* to f by choosing the remaining coefficients as indicated in Figure 3.4(bottom)). ♣

The following three lemmas generalize the procedure worked out in Example 3.1. The first lemma is just a repeated application of constraint $(D2)$.

(3.5) Lemma. *For* $s, l \in \mathbb{N}$, $t \in \mathbb{N} \cup \{0\}$,

$$C(\ldots, s, 0^l, t) = (l+1)C(\ldots, s-1, 0^{l-1}, 1, t) - lC(\ldots, s-1, 0^l, t+1), \qquad (3.6)$$

$$C(t, 0^l, s, \ldots) = (l+1)C(t, 1, 0^{l-1}, s-1, \ldots) - lC(t+1, 0^l, s-1, \ldots).$$

Proof. We prove the first equality – the second equality follows by the symmetric argument. For $l = 1$, (3.6) is just $(D2)$. Applying $(D2)$ in the form

$$C(\ldots, s-1, 0^{l-1}, 1, 0, t) = 2C(\ldots, s-1, 0^{l-1}, 0, 1, t) - C(\ldots, s-1, 0^{l-1}, 0, 0, t+1),$$

to the first right hand side term of

$$C(\ldots, s, 0^l, 0, t) = (l+1)C(\ldots, s-1, 0^{l-1}, 1, 0, t) - lC(\ldots, s-1, 0^l, 1, t)$$

yields

$$C(\ldots, s, 0^{l+1}, t) = (l+2)C(\ldots, s-1, 0^l, 1, t) - (l+1)C(\ldots, s-1, 0^{l+1}, t+1),$$

the induction step from l to $l+1$. ♣

The second lemma shows that in lieu of f on σ, we may consider a degree m-1 polynomial f^* on the reduced simplex σ^* whose vertices have barycentric coordinates

$$(m, 0^{m-1}, 1), \quad (1, 0^{i-1}, m-1, 0^{m-i-1}, 1),\ i \in \{1, \ldots, m-1\} \quad (1, 0^{m-1}, m).$$

The reduction corresponds to enforcing the constraints $(D1)$ and $(D3)$ across the (hyper-) facets that exclude either the vertex 0 or the vertex $\sum_{i=1}^m e_i$.

(3.7) Lemma. *For* $m > 2$, $i \in \{1, \ldots, m-1\}$,

$$C(m+1, 0^m) = C(m, 0^{m-1}, 1)$$

$$C(0^i, m+1, 0^{m-i}) = C(1, 0^{i-1}, m-1, 0^{m-i-1}, 1)$$

$$C(0^m, m+1) = C(1, 0^{m-1}, m).$$

Proof. If $i \le m/2$, then, by Lemma 3.5, $(D2)$, $(D3)$, $(D1)$

$$C(0^i, m+1, 0^{m-i}) = (m-i)C(0^i, m, 0^{m-i-2}, 1, 0) - (m-i-1)C(0^i, m, 0^{m-i-2}, 0, 1)$$
$$= C(0^i, m, 0^{m-i-1}, 0, 1) = C(1, 0^{i-1}, m-1, 0^{m-i-1}, 1)$$

(For $i = 0$, omit the last step.) Similarly, for $i > m/2$,

$$C(0^i, m+1, 0^{m-i}) = (m-i)C(0, 1, 0^{i-2}, m, 0^{m-i}) - (m-i-1)C(1, 0, 0^{i-2}, m, 0^{m-i})$$
$$= C(1, 0^{i-1}, m, 0^{m-i}) = C(1, 0^{i-1}, m-1, 0^{m-i-1}, 1).$$

(For $i = m$, omit the last step.) ♣

Notation. We use $c(x)$ for the coefficients of the reduced polynomial, f^*, over the reduced simplex, σ^*, i.e.

$$c(m-1, 0^m) := C(m, 0^{m-1}, 1)$$
$$c(0^i, m-1, 0^{m-i}) := C(1, 0^{i-1}, m-1, 0^{m-i-1}, 1),\ i \in \{1, \dots, m-1\}$$
$$c(0^m, m-1) := C(1, 0^{m-1}, m).$$

The third lemma generalizes (3.3) by expressing the vertex coefficients of f^* in terms of coefficients of a lower dimensional subsimplex. The main theorem, Theorem 3.11, uses the lemma with $s := m-1$.

(3.8) Lemma. *For $l \in \{0, \dots, s+1\}$, $s \in \mathbb{N}$,*

$$c(0^{s+1-l}, s, 0^l) = (-1)^s \sum_{n=0}^{s} (-1)^n \sum_{\sum_{i=1}^{n+1} j_i = s} p(l, J_n) c(0^{s+1-n}, J_n), \tag{3.9}$$

where $j_i > 0$ for $i = 1, \dots, n$; $j_{n+1} \ge 0$; $J_n := j_{n+1}+1, j_n, \dots, j_2, j_1 - 1$ for $n > 0$; $J_0 := j_1$, and

$$p(l, J_n) := \frac{l}{l-n} \prod_{i=1}^{n+1} (l-i)^{j_i}$$

is a polynomial of degree s in l.

Here $\sum_{i+j+k+l=s}$ stands for $\sum_{i=1}^{s} \sum_{j=1}^{s-i} \sum_{k=1}^{s-i-j}$ and $l := s-i-j-k$ is a slack variable, called j_{n+1} above. Note that $p(l, J_n) = 0$ for $l < n$ and that the right hand side is well-defined also for $l = n$ since the denominator $(l - n)$ cancels with a factor of the numerator. For $l = 0$, the only non-zero summand is the one with $n = 0$.

Proof. We first show that

$$c(\dots, r + s, 0^l, t, \dots) \tag{3.10}$$
$$= (-1)^s l^s c(\dots, r, 0^l, t + s, \dots) - (l + 1) \sum_{j=1}^{s} (-1)^j l^{j-1} c(\dots, r + s - j, 0^{l-1}, 1, t + j - 1, \dots).$$

For $s = 1$, this is Lemma 3.5. By induction,

$$\begin{aligned} c(\dots, r + s + 1, 0^l, t, \dots) =& (-1)^s l^s c(\dots, r + 1, 0^l, t + s, \dots) \\ & -(l + 1) \sum_{j=1}^{s} (-1)^j l^{j-1} c(\dots, r + s + 1 - j, 0^{l-1}, 1, t + j - 1, \dots) \end{aligned}$$

and

$$c(\dots, r + 1, 0^l, t + s, \dots) = (l + 1)c(\dots, r, 0^{l-1}, 1, t + s, \dots) - lc(\dots, r, 0^l, t + s + 1, \dots).$$

A first application of (3.10) yields

$$\begin{aligned} &(-1)^s c(0^{s+1-l}, s, 0^l) \\ &\quad =(-1)^s c(0^{s-l+1}, s, 0^{l-1}, 0) \\ &\quad =(l - 1)^s c(0^{s+1}, s) - l \sum_{j_1=1}^{s} (-1)^{s+j_1} (l - 1)^{j_1 - 1} c(0^{s-l+1}, s - j_1, 0^{l-2}, 1, j_1 - 1). \end{aligned}$$

Again by (3.10),

$$c(0^{s-l+1}, s - j_1, 0^{l-2}, 1, j_1 - 1) = (-1)^{s-j_1} (l - 2)^{s-j_1} c(0^{s-l+1}, 0^{l-1}, 1 + s - j_1, j_1 - 1) + \dots.$$

Thus, repeated application of (3.10) to the coefficients whose index depends on l gives the result:

$$
\begin{aligned}
&(-1)^s c(0^{s+1-l}, s, 0^l)\\
&\quad = \frac{l}{l}(l-1)^s c(0^{s+1}, s) - \sum_{j_1=1}^{s} \frac{l}{l-1}(l-1)^{j_1}(l-2)^{s-j_1} c(0^s, s-j_1+1, j_1-1)\\
&\quad + \sum_{j_1=1}^{s}\sum_{j_2=1}^{s-j_1}(-)^{j_1+j_2}\frac{l}{l-2}(l-1)^{j_1}(l-2)^{j_2} c(0^{s+1-l}, s-j_1-j_2, 0^{l-3}, 1, j_2, j_1-1)\\
&\quad = \ldots\\
&\quad = \sum_{n=0}^{s}(-1)^n \sum_{\sum_{i=1}^{n+1} j_i = s} \frac{l}{l-n}\prod_{i=1}^{n+1}(l-i)^{j_i} c(0^{s-n+1}, J_n).
\end{aligned}
$$

♣

In particular, Lemma 3.8 expresses a vertex coefficient, $c(0^{m-l}, m-1, 0^l)$, as a polynomial $P(l)$ of degree $m-1$ with coefficients independent of l. This is central to the next theorem.

(3.11) Theorem. *There exists a symmetric, shift-invariant, m-variate C^1 function f of degree $m+1$ over Δ iff*

$$
\sum_{\alpha\in\{0,1\}^m} (-)^\alpha f(\alpha) = 0, \qquad \textit{where } (-)^\alpha := (-1)^{\alpha_1}\cdots(-1)^{\alpha_m}, \tag{3.12}
$$

i.e. the alternating sum of the values at the vertices of the unit-cube is zero.

Proof. Using symmetry, Lemma 3.7 and Lemma 3.8,

$$\sum_{\alpha\in\{0,1\}^m} (-)^\alpha f(\alpha)$$

$$= \sum_{l=0}^{m} (-)^l \binom{m}{l} f(1^{m-l}, 0^l).$$

$$= \sum_{l=0}^{m} (-)^l \binom{m}{l} C(0^{m-l}, m+1, 0^l)$$

$$= \sum_{l=0}^{m} (-)^l \binom{m}{l} c(0^{m-l}, m-1, 0^l)$$

$$= \sum_{l=0}^{m} (-)^l \binom{m}{l} P(l).$$

The last expression is the mth divided difference of the polynomial P which is of degree $m-1$. Hence the sum is zero.

To prove sufficiency, we observe that each vertex coefficient of f^* over σ^* occurs in exactly one constraint of type $(D2)$ and this constraint is enforced by (3.12). Hence, even if there is a dependency among the remaining constraints with support only in σ^*, the constraints are homogeneous, hence have a solution. The constraints of type $(D1)$ and $(D3)$ with support outside σ^* are enforced by setting for $r = 0, \ldots, m-1$,

$$C(0, r+1, \ldots) = C(1, r, \ldots)$$

$$C(\ldots, r+1, 0) = C(\ldots, r, 1)$$

$$C(0, r+1, \ldots, 0) = C(1, r, \ldots, 0).$$

The constraints of type $(D2)$ with coefficients on the edge $(0, e_1)$, resp. on the edge $(\sum_{i=1}^{m-1} e_i, \sum_{i=1}^{m} e_i)$, finally, are enforced by setting for $r = 1, \ldots, m-1$,

$$C(m-r, r, 0^{m-1}) = C(m-r, r-1, 1, 0^{m-2})$$

$$C(0^{m-1}, r, m-r) = C(0^{m-2}, 1, r-1, m-r).$$

♣

4. Remarks and conjecture

We now know that m-variate C^1-splines of degree $m+1$ do not interpolate stably. It is also known that univariate C^k splines of degree $k+1$ do not interpolate stably (at breakpoints for k even, at midpoints for k odd). Hence, we hope to fill out the area between these two 'axes', expecting to find that interpolation by C^k m-variate splines of degree $k+m$ is unstable. For k even, the data Δ, V do not seem helpful: due to the symmetry across facets, the interpolant f is C^2 if it is C^1. However, we can show that interpolation by C^2 quartics at the centroids of an equilateral triangulation is unstable.

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