In this paper we present a form of law of large numbers on Pn, the space of n x n symmetric positive-definite (SPD) matrices equipped with Fisher-Rao metric. Specialy, we propose a recursive algorithm for estimating the Karcher expectation of an arbitrary distribution defined on Pn, and we show that the estimates computed by the recursive algorithm asymptotically converge in probability to the correct Karcher expectation. The steps in the recursive algorithm mainly consist of making appropriate moves on geodesics in Pn and the algorithm is simple to implement and it offers a tremendous gain in computation time of several orders of magnitude over existing non-recursive algorithms. We elucidate the connection between the more familiar law of large numbers for real-valued random variables and the asymptotic convergence of the proposed recursive algorithm, and our result provides an example of a new form of law of large numbers for SPD matrices. From a practical viewpoint, the computation of the mean of a collection of SPD matrices is a fundamental ingredient in many algorithms in machine learning, computer vision and medical imaging applications. We report experiments using the proposed recursive algorithm for K-means clustering, demonstrating the algorithm’s efficiency and accuracy.

I. Contributions

- Novel theoretical results on a recursive Karcher expectation estimator (RKEE) for any probability distribution on Pn.
- A proof of the law of large numbers for Pn.
- Convergence to the Karcher expectation in probability as n → ∞.
- Sequentially fed to estimators.

II. Approaches

- Recursive Karcher Expectation Estimator (RKEE)
  - Definition: Let Xn,k = X1, X2, ..., Xn, be i.i.d. random samples drawn from probability measure P(X) on Pn, the RKEE at step k is recursively computed by following equations.
  - M1 = X1
  - Mk = M(k-1)(1/k)XkM(k-1)
  - M̄k = On the geodesic between Mk (last estimation) and Xk, (the new incoming sample), and
  - d(M̄k, Mk) / d(M̄k, Xk) = 1/k
  - If P(X) is a symmetric distribution, i.e. P(X) = P(X|µ) where µ is the expectation of the distribution, RKEE is an unbiased estimator, or E[X] = E[µk], where E[µk] denotes the Karcher expectation.
  - RKEE will converge to the Karcher expectation in probability for any finite variance distributions on Pn.
  - Experiments demonstrate that RKEE has similar accuracy as the non-recursive counterpart, but is much more efficient, especially for online estimation problems.

III. Experiments

- Performance of RKEE
  - i.i.d. samples from Log-GM.
  - Sequentially fed to estimators.
  - Estimation error: squared geodesic distance with the ground truth.
  - 20 trials, mean error/speed.
  - RKEE vs non-recursive Karcher mean.

IV. References


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