Single-Source All-Destinations Shortest Paths With Negative Costs

- Directed weighted graph.
- Edges may have negative cost.
- No cycle whose cost is < 0.
- Find a shortest path from a given source vertex s to each of the n vertices of the digraph.

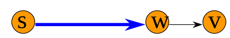
Single-Source All-Destinations Shortest Paths With Negative Costs

- Dijkstra's $O(n^2)$ single-source greedy algorithm doesn't work when there are negative-cost edges.
- Floyd's $\Theta(n^3)$ all-pairs dynamic-programming algorithm does work in this case.

Bellman-Ford Algorithm

- Single-source all-destinations shortest paths in digraphs with negative-cost edges.
- Uses dynamic programming.
- Runs in $O(n^3)$ time when adjacency matrices are used.
- Runs in O(ne) time when adjacency lists are used.

Decision Sequence



- To construct a shortest path from the source to vertex v, decide on the max number of edges on the path and on the vertex that comes just before v.
- Since the digraph has no cycle whose length is < 0, we may limit ourselves to the discovery of cycle-free (acyclic) shortest paths.
- A path that has no cycle has at most n-1 edges.

Problem State

- Problem state is given by (u,k), where u is the destination vertex and k is the max number of edges.
- (v,n-1) is the state in which we want the shortest path to v that has at most n-1 edges.

Cost Function

- Let d(v,k) be the length of a shortest path from the source vertex to vertex v under the constraint that the path has at most k edges.
- d(v,n-1) is the length of a shortest unconstrained path from the source vertex to vertex v.
- We want to determine d(v,n-1) for every vertex v.

Value Of d(*,0)

- d(v,0) is the length of a shortest path from the source vertex to vertex v under the constraint that the path has at most 0 edges.

- d(s,0) = 0.
- d(v,0) = infinity for v != s.

Recurrence For d(*,k), k > 0

- d(v,k) is the length of a shortest path from the source vertex to vertex v under the constraint that the path has at most k edges.
- If this constrained shortest path goes through no edge, then d(v,k) = d(v,0).

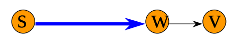
Recurrence For d(*,k), k > 0

- If this constrained shortest path goes through at least one edge, then let w be the vertex just before v on this shortest path (note that w may be s).

- We see that the path from the source to w must be a shortest path from the source vertex to vertex w under the constraint that this path has at most k-1 edges.
- d(v,k) = d(w,k-1) + length of edge (w,v).

Recurrence For d(*,k), k > 0

- d(v,k) = d(w,k-1) + length of edge (w,v).



- We do not know what w is.
- We can assert
  - d(v,k) = min{d(w,k-1) + length of edge (w,v)}, where the min is taken over all w such that (w,v) is an edge of the digraph.
- Combining the two cases considered yields:
  - d(v,k) = min{d(v,0),
    min{d(w,k-1) + length of edge (w,v)}}

Pseudocode To Compute d(*,*)

```
// initialize d(*,0)
d(s,0) = 0;
d(v,0) = infinity, v != s;
// compute d(*,k), 0 < k < n
for (int k = 1; k < n; k++)
{
   d(v,k) = d(v,0), 1 <= v <= n;
   for (each edge (u,v))
      d(v,k) = min{d(v,k), d(u,k-1) + cost(u,v)}
}
```

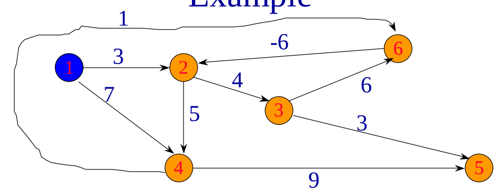
Complexity

- Theta(n) to initialize d(*,0).
- Theta(n^2) to compute d(*,k) for each k > 0 when adjacency matrix is used.
- Theta(e) to compute d(*,k) for each k > 0 when adjacency lasts are used.
- Overall time is Theta(n^3) when adjacency matrix is used.
- Overall time is Theta(ne) when adjacency lists are used.
- Theta(n^2) space needed for d(*,*).

p(*,*)

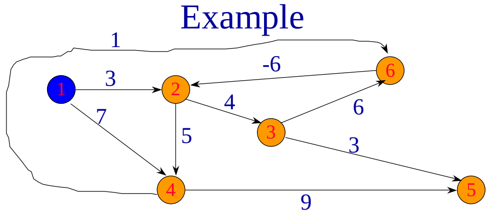
- Let p(v,k) be the vertex just before vertex v on the shortest path for d(v,k).
- p(v,0) is undefined.
- Used to construct shortest paths.

Example



Source vertex is 1.

Example



k \ v	1	2	3	4	5	6
0	0	-	-	-	-	-
1	0	3	-	7	-	-
2	0	3	7	7	16	8
3	0	2	7	7	10	8
4	0	2	6	7	10	8

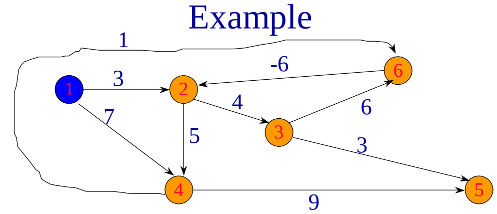
d(v,k)

v →, k ↓

-	-	-	-	-	-
-	1	-	1	-	-
-	1	2	1	4	4
-	6	2	1	3	4
-	6	2	1	3	4

p(v.k)

Example



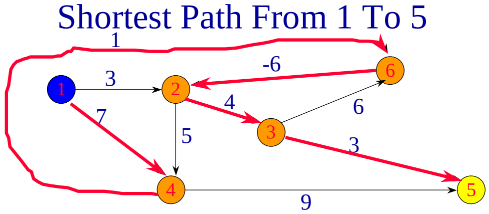
k \ v	1	2	3	4	5	6
4	0	2	6	7	10	8
5	0	2	6	7	9	8

v →, k ↓

-	6	2	1	3	4
-	6	2	1	3	4

d(v,k) p(v.k)

Shortest Path From 1 To 5



	1	2	3	4	5	6
5	0	2	6	7	9	8

d(v,5)

1	2	3	4	5	6
-	6	2	1	3	4

p(v,5)

Observations

- d(v,k) = min{d(v,0),
  min{d(w,k-1) + length of edge (w,v)}}
- d(s,k) = 0 for all k.
- If d(v,k) = d(v,k-1) for all v, then d(v,j) = d(v,k-1), for all j >= k-1 and all v.
- If we stop computing as soon as we have a d(*,k) that is identical to d(*,k-1) the run time becomes
  - $O(n^3)$ when adjacency matrix is used.
  - O(ne) when adjacency lists are used.

Observations

- The computation may be done in-place.

  d(v) = min{d(v), min{d(w) + length of edge (w,v)}}

  instead of

  d(v,k) = min{d(v,0),
  min{d(w,k-1) + length of edge (w,v)}}
- Following iteration k, d(v,k+1) <= d(v) <= d(v,k)
- On termination d(v) = d(v,n-1).
- Space requirement becomes O(n) for d(*) and p(*).