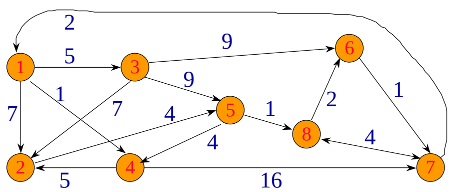
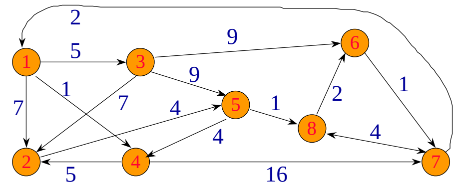
# All-Pairs Shortest Paths

- Given an n-vertex directed weighted graph, find a shortest path from vertex i to vertex j for each of the $n^2$ vertex pairs (i,j).

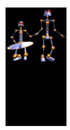


# Dijkstra's Single Source Algorithm

- Use Dijkstra's algorithm n times, once with each of the n vertices as the source vertex.



# Performance

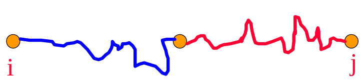


- Time complexity is $O(n^3)$ time.
- Works only when no edge has a cost < 0.

# Dynamic Programming Solution

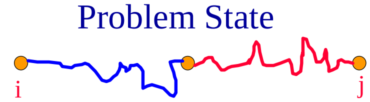
- Time complexity is Theta($n^3$) time.
- Works so long as there is no cycle whose length is < 0.
- When there is a cycle whose length is < 0, some shortest paths aren't finite.
  - If vertex 1 is on a cycle whose length is -2, each time you go around this cycle once you get a 1 to 1 path that is 2 units shorter than the previous one.
- Simpler to code, smaller overheads.
- Known as Floyd's shortest paths algorithm.

# Decision Sequence



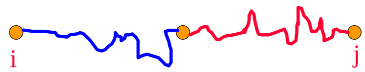

- First decide the highest intermediate vertex (i.e., largest vertex number) on the shortest path from i to j.
- If the shortest path is i, 2, 6, 3, 8, 5, 7, j the first decision is that vertex 8 is an intermediate vertex on the shortest path and no intermediate vertex is larger than 8.
- Then decide the highest intermediate vertex on the path from i to 8, and so on.

# Problem State




- (i,j,k) denotes the problem of finding the shortest path from vertex i to vertex j that has no intermediate vertex larger than k.
- (i,j,n) denotes the problem of finding the shortest path from vertex i to vertex j (with no restrictions on intermediate vertices).

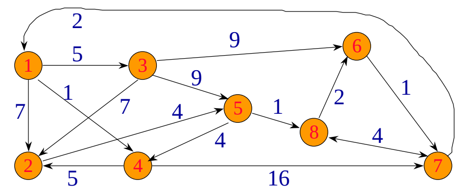
## Cost Function




- Let c(i,j,k) be the length of a shortest path from vertex i to vertex j that has no intermediate vertex larger than k.

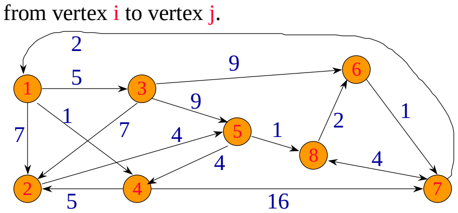
## c(i,j,n)

- c(i,j,n) is the length of a shortest path from vertex i to vertex j that has no intermediate vertex larger than n.
- No vertex is larger than n.
- Therefore, c(i,j,n) is the length of a shortest path from vertex i to vertex j.



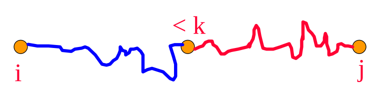

## c(i,j,0)

- c(i,j,0) is the length of a shortest path from vertex i to vertex j that has no intermediate vertex larger than 0.
  - Every vertex is larger than 0.
  - Therefore, c(i,j,0) is the length of a single-edge path from vertex i to vertex j.



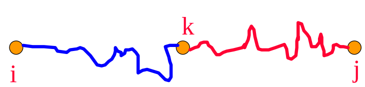

## Recurrence For c(i,j,k), k > 0

- The shortest path from vertex i to vertex j that has no intermediate vertex larger than k may or may not go through vertex k.
- If this shortest path does not go through vertex k, the largest permissible intermediate vertex is k-1. So the path length is c(i,j,k-1).



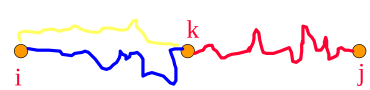

## Recurrence For c(i,j,k) ), k > 0

- Shortest path goes through vertex k.



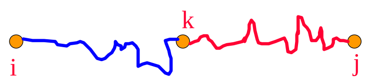

- We may assume that vertex k is not repeated because no cycle has negative length.
- Largest permissible intermediate vertex on i to k and k to j paths is k-1.

## Recurrence For c(i,j,k) ), k > 0



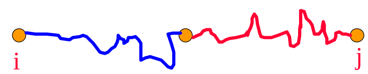

- i to k path must be a shortest i to k path that goes through no vertex larger than k-1.
- If not, replace current i to k path with a shorter i to k path to get an even shorter i to j path.

## Recurrence For c(i,j,k) ), k > 0




- Similarly, k to j path must be a shortest k to j path that goes through no vertex larger than k-1.
- Therefore, length of i to k path is c(i,k,k-1), and length of k to j path is c(k,j,k-1).
- So, c(i,j,k) = c(i,k,k-1) + c(k,j,k-1).

## Recurrence For c(i,j,k) ), k > 0




- Combining the two equations for c(i,j,k), we get c(i,j,k) = min{c(i,j,k-1), c(i,k,k-1) + c(k,j,k-1)}.
- We may compute the c(i,j,k)s in the order k = 1, 2, 3, …, n.

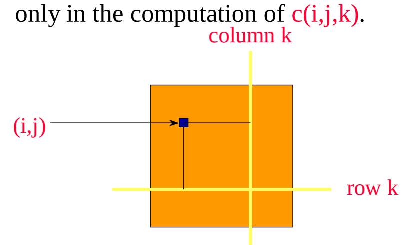
## Floyd's Shortest Paths Algorithm

```
for (int k = 1; k <= n; k++)
  for (int i = 1; i <= n; i++)
    for (int j = 1; j <= n; j++)
       c(i,j,k) = min{c(i,j,k-1),
                      c(i,k,k-1) + c(k,j,k-1)};
```

- Time complexity is $O(n^3)$.
- More precisely $Theta(n^3)$.
- $Theta(n^3)$ space is needed for c(*,*,*).

## Space Reduction

- c(i,j,k) = min{c(i,j,k-1), c(i,k,k-1) + c(k,j,k-1)}
- When neither i nor j equals k, c(i,j,k-1) is used only in the computation of c(i,j,k).




- So c(i,j,k) can overwrite c(i,j,k-1).

## Space Reduction

- c(i,j,k) = min{c(i,j,k-1), c(i,k,k-1) + c(k,j,k-1)}
- When i equals k, c(i,j,k-1) equals c(i,j,k).
  - c(k,j,k) = min{c(k,j,k-1), c(k,k,k-1) + c(k,j,k-1)}
    = min{c(k,j,k-1), 0 + c(k,j,k-1)}
    = c(k,j,k-1)
- So, when i equals k, c(i,j,k) can overwrite c(i,j,k-1).
- Similarly when j equals k, c(i,j,k) can overwrite c(i,j,k-1).
- So, in all cases c(i,j,k) can overwrite c(i,j,k-1).

## Floyd's Shortest Paths Algorithm

```
for (int k = 1; k <= n; k++)
  for (int i = 1; i <= n; i++)
    for (int j = 1; j <= n; j++)
       c(i,j) = min{c(i,j), c(i,k) + c(k,j)};
```

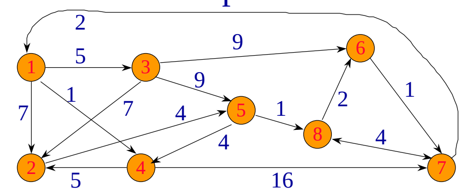
- Initially, c(i,j) = c(i,j,0).
- Upon termination, c(i,j) = c(i,j,n).
- Time complexity is $Theta(n^3)$.
- $Theta(n^2)$ space is needed for c(*,*).

## Building The Shortest Paths

- Let kay(i,j) be the largest vertex on the shortest path from i to j.
- Initially, kay(i,j) = 0 (shortest path has no intermediate vertex).

```
for (int k = 1; k <= n; k++)
  for (int i = 1; i <= n; i++)
    for (int j = 1; j <= n; j++)
      if (c(i,j) > c(i,k) + c(k,j))
      {kay(i,j) = k; c(i,j) = c(i,k) + c(k,j);}
```

## Example




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| - | 7 | 5 | 1 | - | - | - | - |
| - | - | - | - | 4 | - | - | - |
| - | 7 | - | - | 9 | 9 | - | - |
| - | 5 | - | - | - | - | 16 | - |
| - | - | - | 4 | - | - | - | 1 |
| - | - | - | - | - | - | 1 | - |
| 2 | - | - | - | - | - | - | 4 |
| - | - | - | - | - | 2 | 4 | - |

Initial Cost Matrix
c(*,*) = c(*,*,0)

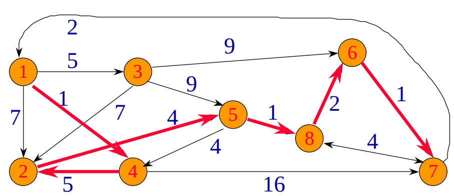
## Final Cost Matrix c(*,*) = c(*,*,n)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 6 | 5 | 1 | 10 | 13 | 14 | 11 |
| 10 | 0 | 15 | 8 | 4 | 7 | 8 | 5 |
| 12 | 7 | 0 | 13 | 9 | 9 | 10 | 10 |
| 15 | 5 | 20 | 0 | 9 | 12 | 13 | 10 |
| 6 | 9 | 11 | 4 | 0 | 3 | 4 | 1 |
| 3 | 9 | 8 | 4 | 13 | 0 | 1 | 5 |
| 2 | 8 | 7 | 3 | 12 | 6 | 0 | 4 |
| 5 | 11 | 10 | 6 | 15 | 2 | 3 | 0 |

## kay Matrix

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 4 | 0 | 0 | 4 | 8 | 8 | 5 |
| 8 | 0 | 8 | 5 | 0 | 8 | 8 | 5 |
| 7 | 0 | 0 | 5 | 0 | 0 | 6 | 5 |
| 8 | 0 | 8 | 0 | 2 | 8 | 8 | 5 |
| 8 | 4 | 8 | 0 | 0 | 8 | 8 | 0 |
| 7 | 7 | 7 | 7 | 7 | 0 | 0 | 7 |
| 0 | 4 | 1 | 1 | 4 | 8 | 0 | 0 |
| 7 | 7 | 7 | 7 | 7 | 0 | 6 | 0 |

## Shortest Path




Shortest path from 1 to 7.
Path length is 14.

## Build A Shortest Path

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 4 | 0 | 0 | 4 | 8 | 8 | 5 |
| 8 | 0 | 8 | 5 | 0 | 8 | 8 | 5 |
| 7 | 0 | 0 | 5 | 0 | 0 | 6 | 5 |
| 8 | 0 | 8 | 0 | 2 | 8 | 8 | 5 |
| 8 | 4 | 8 | 0 | 0 | 8 | 8 | 0 |
| 7 | 7 | 7 | 7 | 7 | 0 | 0 | 7 |
| 0 | 4 | 1 | 1 | 4 | 8 | 0 | 0 |
| 7 | 7 | 7 | 7 | 7 | 0 | 6 | 0 |

- The path is 1 4 2 5 8 6 7.
- kay(1,7) = 8

  1 → 8 → 7
- kay(1,8) = 5

  1 → 5 → 8 → 7
- kay(1,5) = 4

  1 → 4 → 5 → 8 → 7

## Build A Shortest Path

0 4 0 0 4 8 8 5
8 0 8 5 0 8 8 5
7 0 0 5 0 0 6 5
8 0 8 0 2 8 8 5
8 4 8 0 0 8 8 0
7 7 7 7 7 0 0 7
0 4 1 1 4 8 0 0
7 7 7 7 7 0 6 0

- The path is 1 4 2 5 8 6 7.

  1 → 4 → 5 → 8 → 7

- kay(1,4) = 0

  1 4 → 5 → 8 → 7

- kay(4,5) = 2

  1 4 → 2 → 5 → 8 → 7

- kay(4,2) = 0

  1 4 2 → 5 → 8 → 7

## Build A Shortest Path

0 4 0 0 4 8 8 5
8 0 8 5 0 8 8 5
7 0 0 5 0 0 6 5
8 0 8 0 2 8 8 5
8 4 8 0 0 8 8 0
7 7 7 7 7 0 0 7
0 4 1 1 4 8 0 0
7 7 7 7 7 0 6 0

- The path is 1 4 2 5 8 6 7.

  1 4 2 → 5 → 8 → 7

- kay(2,5) = 0

  1 4 2 5 → 8 → 7

- kay(5,8) = 0

  1 4 2 5 8 → 7

- kay(8,7) = 6

  1 4 2 5 8 → 6 → 7

## Build A Shortest Path

0 4 0 0 4 8 8 5
8 0 8 5 0 8 8 5
7 0 0 5 0 0 6 5
8 0 8 0 2 8 8 5
8 4 8 0 0 8 8 0
7 7 7 7 7 0 0 7
0 4 1 1 4 8 0 0
7 7 7 7 7 0 6 0

- The path is 1 4 2 5 8 6 7.

  1 4 2 5 8 → 6 → 7

- kay(8,6) = 0

  1 4 2 5 8 6 → 7

- kay(6,7) = 0

  1 4 2 5 8 6 7

## Output A Shortest Path

```
public static void outputPath(int i, int j)
{// does not output first vertex (i) on path
  if (i == j) return;
  if (kay[i][j] == 0)  // no intermediate vertices on path
     System.out.print(j + " ");
  else {// kay[i][j] is an intermediate vertex on the path
          outputPath(i, kay[i][j]);
          outputPath(kay[i][j], j);
       }
}
```

## Time Complexity Of outputPath

O(number of vertices on shortest path)